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# Reconsidering and Extending the Conventional/Demographic and *LPE* Models: The *LPd* and *LPi* Restricted Markov Models

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## I. Introduction

This paper considers the old conventional worklife model for workers beginning active (*CVa*) of the Bureau of Labor Statistics (BLS) by deriving and analyzing with new methods its probability mass functions. The *CVa's* median years of additional labor market activity, or equivalently years until final separation, is shown in special cases to be consistent with the Hunt-Pickersgill-Rutemiller (HPR, 1997, 1997a, 1999, 2001) computation. Like the Markov model which replaced it, the *CVa* completely describes sample path behavior of workers between the active and inactive states. We then discuss its variant which covers all workers, grouping those starting active and inactive, *CV*. The *CV* model merits the names demographic or actuarial model; we discuss our preference for these terms instead of Richards' (1999, 2000) use of the term "conventional model." Probability mass functions for the demographic model are explored and difficulties noted. We then map the *CVa* into a restricted Markov model, extending earlier work by Skoog (2002). We find two absorbing states, death and inactivity, and call this the *LPd* model, because it captures the *L* ("living") and *P* ("participation") components of the so-called *LPE* model, while imposing strong dependence ("*d*") on the probability of future activity given the past activity or inactivity status and survival. This model is estimated and its mass functions are computed. Comparisons and connections with the *CVa* and *CV*/demographic models are noted. We present the machinery to analyze all of the statistical properties of the *LPd* distributions' years of additional activity, using our earlier results, by proving that these models are also Markov models, with highly restricted transition properties.

We do the same for the new *LPi* model ("*i*" for independence with respect to initial labor force state) which provides a natural probabilistic foundation for the *LPE* model. Despite these analytical and theoretical extensions and estimates for the true conventional model, as extended to the *LPd* model, the *CVa* model implicit in HPR's computation, the *CV*/demographic/actuarial/Richards variant, and the *LPi* model, all embody but two parameters for each age, are thus nested within the three parameter Markov model, and rejected by the data. Consequently we are presently left with the more general Markov model for analysis of years of activity and years to final separation.

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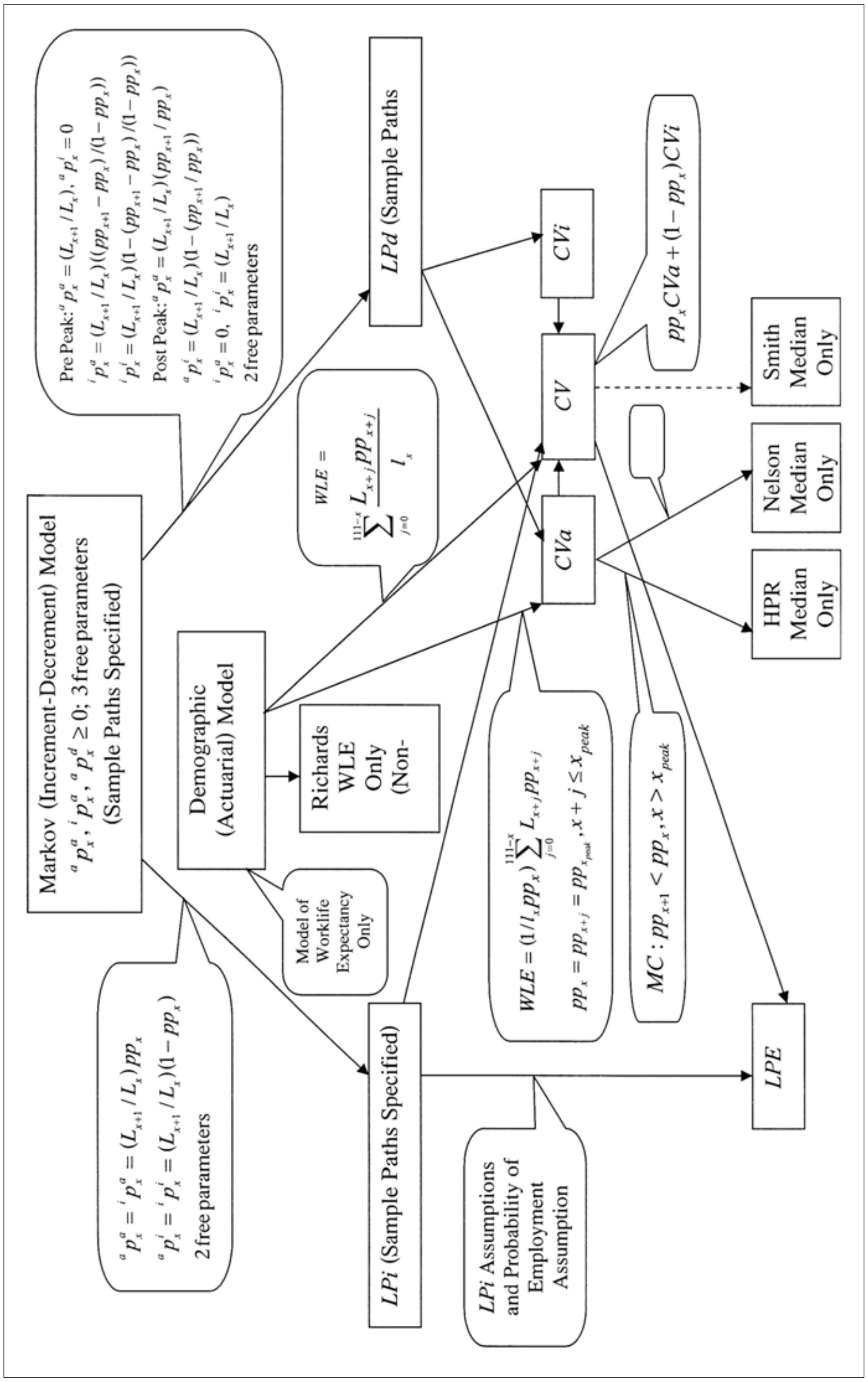


Figure 1
Relationships Among the Various Models

Figure 1 illustrates some of the basic relationships among the models discussed in this paper. At the top, the Markov (Increment-Decrement) Model imposes no restrictions on transition probabilities other than

$$ {}^{a}p_x^{a}, {}^{a}p_x^{i}, {}^{i}p_x^{a}, {}^{i}p_x^{i} \geq 0 \text{ , } x < TA-1;\ {}^{a}p_x^{a} + {}^{a}p_x^{i} + {}^{a}p_x^{d} = 1 \text{ , } {}^{i}p_x^{a} + {}^{i}p_x^{i} + {}^{i}p_x^{d} = 1 \text{ ,} $$
$$ {}^{a}p_x^{d} = {}^{i}p_x^{d} \text{ for all } x; $$

and

$$ {}^{a}p_x^{a} = {}^{a}p_x^{i} = {}^{i}p_x^{a} = {}^{i}p_x^{i} = 0 \text{ , } x \geq TA-1, $$

for some truncation age *TA*. Our *LPd* and *LPi* models are special cases of the Markov (Increment-Decrement) model defined by assigning specific values to transition probabilities. The *LPd* model yields the conventional model for actives *CVa*, faithful to BLS *Bulletin 2135* (1982), as well as a model for inactives, *CVi,* that heretofore was unarticulated but was implicit in the conventional model. The conventional model regardless of labor force status, *CV,* is a weighted average of *CVa* and *CVi*. The HPR, Smith (undated) and Nelson (1983) estimators for the median years to labor force separation result when *CVa* is restricted to ages at and beyond peak labor force participation and participation rate data fulfill a monotonicity condition. The *LPi* model provides a probabilistic foundation for the life and participation part of the *LPE* model. Viewed from the bottom and moving to the top of Figure 1, each model is a special case or result, involving age and special transition probability assumptions, of the model above it. Thus, all of the models are Markov models but with fewer restrictions as we move to the top of the figure. Figure 1 also shows demographic models of labor force activity as promulgated in *Bulletin 2135*. These models are not probabilistic, but the means of the *LPd* and *LPi* models for years of activity closely agree with the worklives produced by those models.

## II. Analytical Approach and Terminology

### A. General Commentary

Before proceeding to the models individually, it is useful to remark that both the Markov and conventional model employed probabilistic concepts (transition probabilities with embedded mortality probabilities in the former, and mortality and participation probabilities in the latter) and applied them to determine various mathematical expectations. Both were formulated as computations of expected or average years of labor force activity, the former conditional on initially being active or inactive, and the latter unconditionally with respect to initial status. The Markov model thus additionally permitted nontrivial conditioning on initial inactivity. Research on neither model, before our recent work with the Markov model, thoroughly exploited these models' probabilistic implications. We provide this analysis now for the first time by fully articulating what was described by the BLS as the conventional model; we call this the *LPd* model. Closely related are variants of the conventional model, variously called the *CV* (for *conventional*) model*,* the *actuarial,* and/or the *demographic* model. Except for our work, there has been no discussion of sample paths, i.e., the statistical distribution of functions of statuses several years into the future, conditional on current status. Once a model is analyzed and/or augmented so as to permit the specification of such general distribution functions, one has broken through the barrier that had restricted previous study to expectations. Our theoretical framework permits the study of natural random variables, such as *YA*, years of additional activity, and *YFS*, years to final separation.[1] In each model, any statistic involving these random variables, such as the mean, median, or mode, may now be studied in the population or in a sample. We may thus compare an estimator claimed to be estimating the "median" with proper estimators of the median, within the context of any particular model.[2] We may alternatively start with an estimator, or an estimating equation, and locate a model in which it is justified, as we do with the HPR estimator.

Finally, it is useful to elaborate further on probabilistic underpinnings which have been absent or neglected in this literature in all of the models from the vantage point of stochastic process theory. Let $t$ denote age (we will also use $x$ below, where in agreement with common usage) and let $X_{t+j}$ be a random variable taking on one of the "status" values "*a*" "*i*" or "*d*" at age *t+j*. We need to understand or specify what each model says or can reasonably be expected to say about joint conditional probabilities such as

$$P\left(X_{t+1},\ldots,X_{111} \mid X_t = a\right),^3$$

$P\left(X_{t+1},\ldots,X_{111} \mid X_t = i\right)$ and the unconditional probabilities such as

$$P\left(X_{t+1},\ldots,X_{111}\right).$$

Here, 111 is taken to be an age by which everyone has died.

Once one dies, one never becomes active, so these joint probabilities will not generally factor,[4] i.e., the joint conditional and unconditional distributions are not independent. The Markov model induces these probabilities; other worklife models may need extension so that these joint probabilities are well

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[1]These references to random variables (in italics, and using all capital letters) are generic. We will need a notation which also reflects the starting age and activity status.

[2]For example, the HPR estimator is claimed to estimate the "median" of the years to final separation random variable, *YFS*, within the Markov model, but it is an inappropriate estimator for this model (it is not statistically consistent). The same estimator could be motivated by another means, what we call an *LPi* model, but it is again ill suited to that model. It is shown in this paper to be consistent instead for only the *LPd* median for certain ages, and in the presence of regularity conditions not present in their data.

[3]These are, with finitely many time periods, tail $\sigma$-fields, involving the entire future of the process. That is, we need a probability measure that maps active, inactive and death events, their complements, unions, and intersections into the interval [0,1].

[4]A partial factorization defines our *LPi* model in Section VI; but, because the distribution of $X_t$ is different if $X_t = d$ than if $X_t = a$ or $X_t = i$, the $X_t$ random variables cannot be strictly statistically independent.

defined for them as well. Once such conditional probabilities are determined, random variables such as

$$YA_{t,a} \equiv \sum_{i=1}^{111\text{-}t} \mathrm{I}_{\left[X_{t+i}=a|X_t=a\right]},$$

which counts the *s* years of additional activity after age *t* starting active, would have well specified probabilities $p_{YA}(t,a,s)$ for $s = .5, 1.5, \ldots, 110.5 - t$. In this definition the symbol "I" denotes the "indicator function" equaling one when the subscripted event occurs. This approach has been developed in Skoog and Ciecka (SC, 2001, 2001a, and 2002).

Similarly, for any number $k_t$ of years into the future, the probability of any sample path event, such as

$$\mathrm{P}\left[X_{t+k_t+j+1} = i \text{ or } d,\ X_{t+k_t} = a,\ j = 0,1,\ldots \mid X_t = a\right]$$

is well defined, allowing us to compute the probabilities $p_{YFS}(t,a,s)$ of final separation as occurring *s* years into the future, for a person age *t* and active, for $s = 0,1,\ldots 111 - t$. The theory and tables based on this idea have been presented in SC (2003). Without a specification of these joint probabilities, notions such as years to final separation are not well defined − underidentified in econometric parlance. Underidentified parameters cannot in principle be consistently estimated.

In Section III we turn to a thorough analysis of the conventional model and its variants, permitting its probability mass functions to be studied. We develop the probability mass functions for the *YA* and *YFS* random variables, years of activity and years until final separation, for the several models which are closely related to each other, and are sometimes thought to be competitors of the Markov model. In fact, these models will be seen to be very special cases of the Markov model, although not heretofore appreciated as such. Our guide as to what the conventional model was historically is taken from Appendix B of *Bulletin 2135*, in which the BLS already had earlier, in Chapter 3, introduced the increment-decrement model. The reader seeking additional detail is encouraged to review this appendix.

### B. Models, Random Variables within Models, Population Characteristics of Random Variables, Estimators, and Recipes

We have chosen the words in this section title very carefully, in our attempt to summarize, synthesize, and extend a worklife literature that has often not been careful about these distinctions. By a model we mean a specification sufficiently detailed so as to imply the probability laws for the labor force status of individuals over time. The model may be, like the Markov model, formal, complete and mathematically rigorous. It may be informal and verbal, but complete, as has been the conventional model starting active. It may be informal, verbal and, if not complete, vague but suggestive of a completion, as is the model we describe as the conventional model. Within a model, there are natural random variables of interest, such as *YA* and *YFS*, among others. These random variables have characteristics, such as probability mass functions, means, medians and modes, etc. These characteristics generally possess estimators, once a sampling mechanism according to which data are generated has been specified. Finally, there are recipes − formulae for cooking or preparing something (data). Estimators are recipes capable of being evaluated within the context of a model; a recipe may be proposed without the benefit of a model, in which case it cannot be assessed without model completion. A first example of a recipe is equation (4) below; without more specification, we call this an actuarial or demographic estimator. Another recipe is HPR's equation (18 below). We make these distinctions because, at times, recipes have been confused with estimators (the latter cannot exist outside a model); worse, recipes are sometimes confused with or identified with the models or population parameters themselves.

### III. Demographic Interpretation of Conventional Models *CV* and *CVa*

Consider $l_x$ individuals with fixed socio-economic characteristics, all at exact age *x,* who are alive. The usual actuarial approach to *life expectancy* begins with either the age specific mortality rates $q_x \equiv (l_x - l_{x+1})/l_x$ or the life table $l_{x+1}$, $l_{x+2}$,...—the survivor sequence itself. Associated with the life table sequence $\{l_x\}$ is a smoothed version, $\{L_x\}$, which depicts both the number of *x*-year-olds alive at the midpoint of the interval and the number of years of life between ages *x* and *x*+1 experienced by the $l_x$ persons who have survived to age *x*. The objects $l_x$ and $L_x$ are viewed demographically in two distinct ways: longitudinally (as a time series) they refer to a cohort as it ages, while cross sectionally, at a point in time, they are characteristics of a synthetic stationary population, into which are born the same number, $l_0$ of individuals per year. The radix value $l_0$ of 100,000 is arbitrary but traditional, and leads to $l_{16}$ =97,823 (in BLS *Bulletin 2254*, 1986, p. 11) as the number surviving to age $x = 16$, for example.

Economy wide data on age-specific labor force activity rates (called $w_x$ in *Bulletin 2135*), are denoted by $pp_x$ (for participation at age *x*). These permit the definitions (*cf.* $lw_x$ in *op. cit.*, p. 47) of two labor force functions, analogous to the life table functions $l_x$ and $L_x$

(1) $$lp_x \equiv l_x pp_x$$

(2) $$Lp_x \equiv [(l_x + l_{x+1})/2]pp_x \equiv L_x pp_x$$

What appears to be an asymmetric treatment in the right hand side of (2) is resolved by interpreting $pp_x$ as the midyear participation probability. Alternatively $pp_x$ may be regarded as the proportion of the *x*-year olds who are active at any point within the age (*x, x*+1) year interval, and the portion of a year spent in the labor force on average (Assumption 6).[5] The conventional model then assumes that "there is no turnover of male workers. Every man who enters the labor force does so only once, remaining continuously active from entry until permanent retirement or death" (Assumption 7.) Finally, the conventional model deals with the initial influx of young workers who were

[5]The numbers following a capitalized Assumption are those of *Bulletin 2135*.

initially inactive by assuming that this movement is unidirectional: "prior to the age of peak labor force involvement, men enter but do not voluntarily withdraw. A few die. After that age, workers retire or die, but none reenters the job market" (Assumption 8). The conventional model is different for women, requiring several paragraphs and new concepts to allow temporary or permanent withdrawals for marriage, the birth of a first child, retirement, and death (*op. cit.*, p. 49). The working life table construction continues to mimic the actuarial or demographic life table or recipe by accumulating $Lp_x$ over all ages at and beyond $x$ to obtain $Tw_x$:

$$Tw_x \equiv \sum_{j=0}^{111-x} Lp_{x+j} = \sum_{j=0}^{111-x} L_{x+j}pp_{x+j} = \sum_{j=0}^{111-x} \frac{l_{x+j} + l_{x+j+1}}{2} pp_{x+j} \ . \tag{3}$$

Dividing $Tw_x$ first by $l_x$ and then by $lp_x$ gives, at this point, a recipe for worklife expectancy $ew_x$, for the overall population, and a recipe, $ew'_x$, for the worklife expectancy of the active population, respectively (cf. (23) and (24) in *Bulletin 2135, op. cit.*) as:

$$ew_x = \frac{Tw_x}{l_x} \equiv WLE(CV_x) \tag{4}$$

$$ew'_x = \frac{Tw_x}{lp_x} \equiv WLE(CVa_x), \ x \geq x_{peak} \tag{5}$$

We continue with our discussion of the working life table for men, and define the participation at the age of peak participation, $x_{peak}$, to be $pp_{x_{peak}}$. In *Bulletin 2135*, the peak occurred at age 34. Observing that the $Tw_x$ additional years of total activity are contributed by the *closed population* of the entire population of the $l_x$ people alive at $x$, it is apparent that the calculation of (4) makes sense for any age, before, at, or after the peak. While we might refer to this equation as the demographic or actuarial "model," understanding that it applies *only* to the entire population, such usage generously anticipates some completion, of which two are described below; "actuarial recipe" and "demographic recipe" are more appropriate descriptors at this point. Indeed, Richards takes (4) to be the "conventional model," ignoring the steps necessary to obtain the conventional model for *active* men in (5) and as it needs to be extended for earlier ages, ignoring the BLS conventional model adaptations for women, and ignoring estimator/model issues. Thus, Richards' "conventional model" agrees with the BLS' conventional model for the population as a whole (for men), and does not attempt to define worklife for the *active* population, a major drawback for forensic work, in our view.[6] When he applies the same

[6]His claims for the superiority of the "conventional model" to the Markov model can only be with respect to the Markov model's worklife for the entire economy, defined below as ${}^{\bullet}e_x^a$, something that almost never arises in practice because we will know whether the initial status was *active* or *inactive*, in most cases. Indeed, in the latest and most complete Markov tables, SC (2001a) do not even bother to compute ${}^{\bullet}e_x^a$, but use of equation (31) below would permit this.

equation, (4), to women, his terminology deviates further from the BLS definition of the conventional model. The BLS conventional model for women appears not to have been fully described for at least 48 years. In *Bulletin 2135*, on page 49, appear two equations in that model depicting accessions for women following loss of a husband (21) and separations due to childbirth (22). The reader is referred to the BLS *Bulletin 1204* (Garfinkel, 1957) for the other formulae, which specify other elements missing in these equations—activity rates of women with husbands present, separation rates due to childbirth, and birth rates for married women who never become mothers being a few. For these reasons, we prefer a more historically accurate descriptor for (4) such as the *demographic model* or the *actuarial model*, both of which we shall employ.

The interpretation of the right hand side of (5), which provides an estimator for what we call *CVa*, and what the BLS called the conventional model for the active population, is more problematic. These $Tw_x$ allocations make sense for ages $x \geq x_{peak} = 34$ since the model assumptions imply that, for these ages, everyone who ever will be active is active. Hence the $Tw_x$ additional years of total activity are contributed by the *closed population*[7] of the $lp_x = l_x pp_x$ people alive and active at that age. However, for "young" ages *x,* below that of peak participation, the denominator, $lp_x$ will increase with *x,* reflecting an influx of labor market participants, and so neither numerator nor denominator can refer to *any* closed population; provided that the modest deaths do not exceed the inflow caused by rising $pp_x$, $lp_x$ will be increasing. To maintain a definition of $ew'_x$, the BLS went to great lengths, by modifying both the numerator and denominator in (5), so that it could be applied to these earlier ages.[8] The numerator modification assumed that $pp_x$ equaled $pp_{x_{peak}}$ for ages $x < x_{peak}$ and applied the $Tw_x$ formula to these earlier years embodying this modification. The BLS described this process as including "additional person years of labor force attachment, 'work-years' which don't really occur" (*op. cit.*, p. 51) dubbed "fictitious work-time" (*op. cit.*, p. 57). The denominator modification involved running the life table backwards, so as to give a population which, subject to mortality, would have resulted in the number of participants observed in the labor force at the peak. This admittedly ad hoc construction provides the "closed labor force" in the sense of the BLS conventional model for actives. When the model behind equation (5) is extended in this way to the pre-peak ages, we understand this to be the conventional model for actives, and continue to use the symbol *CVa*.

By contrast, the actuarial or demographic recipe (or, more loosely, "model") applying only to the general population does not appear to require the heroic *CVa* measures. The BLS estimated its conventional model for the last time in *Bulletin 2135*, Appendix B using 1977 men, reporting side-by-side on page 55 in columns (10)-(14) the actuarial/demographic model and in columns (15)-(19)

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[7]The fact that the population is closed for these ages follows from Assumptions 7 and 8.

[8]See the treatment on pages 46 and 47 of *Bulletin 2135* for men, especially Figure B-2, and pages 49-51, especially Figures B-3 to B-6, p. 51.

the *CVa* model. In fact, columns (10)-(14) also depict worklife expectancy computed as a weighted average of $WLE(CVa)$ and $WLE(CVi)$, where the latter, as the development below shows, is taken to be simply 0. Although the BLS did not bother to spell out the conventional model starting inactive, at and beyond the peak age of labor force participation, evidently zero additional years of labor force activity are attained, so that we may write this as $CVi_x \equiv 0$ for $x \geq x_{peak}$. This notation conveys the idea that starting inactive at these ages, the probability mass function is degenerate, concentrated on zero. Nevertheless, there are multiple ways that these *CVa* and *CVi* models may be combined to produce a model of worklife for the entire population which possesses the same worklife expectancy. Although the models have the same worklife expectancy, they are very different in other important economic dimensions, and illustrate key differences between the models, some of which continue to be used by forensic economists. The basic insight goes back to Mincer (1966, p. 102) and Ben-Porath (1973), and notes that an average participation rate of (say) 40% does not mean that 40% of the individuals are almost always in the labor force while 60% are in almost never—a phenomenon since referred to as extreme heterogeneity. Mincer noted that "It means rather that the same individuals are sometimes in and sometimes out during a period of years."—what has been termed since as homogeneity.

Starting with the case of heterogeneity, the *CVa* model is sufficient, as we show below at equation (9), to induce the distribution of additional years in the labor force. *CVi* trivially induces the point mass at 0 years distribution, at least at and beyond peak participation. If the population consists of $pp_x$ of those who follow $CVa_x$ and $1 - pp_x$ of those who follow $CVi_x$, the distribution or mass function for the economy follows the mixture distribution with these weights. We could denote this version of the conventional model symbolically as $CV_x = pp_x CVa_x + (1 - pp_x) CVi_x$. For the pre-peak ages, also present in the BLS table, the same mixture distribution argument would apply to however $CVa_x$ and $CVi_x$ were extended back to earlier ages.

At the other extreme of complete homogeneity, we might assume that each individual becomes active at age $x$ with probability $pp_x$ regardless of employment history. This is the *LPi* model discussed below, and implicit in the LPE model. Worklife expectancy in this model, because of our midpoint dating convention, is ½ of a year greater for actives than inactives; for the economy as a whole, the $pp_x$, $1 - pp_x$ weighted average, taking the ½ into account, essentially reproduces demographic/actuarial/CV worklife expectancies. This is a second mean preserving completion of the mean-oriented basic specification of the conventional model consistent with equation (5).

It happens that neither of these completions corresponded to the BLS' descriptions of the extensions necessary for the conventional model. While they were also not interested in obtaining probability mass functions for years of additional activity, their verbal description, when put in the form of equations (26)-(29) of Section V, constitutes a third extension. As we show, this turns out to be a Markov model, replicates the worklife for the special cases of those starting active, starting inactive, and for the economy as a whole.

With the reasoning of the previous paragraphs in mind, we identify the actuarial/demographic model with the *CV* or conventional model when referring to the entire population, and especially to means, while distinguishing and sharply differentiating these terms when referring to the completely stochastically specified and more elaborate *LPd* model (developed in Sections IV and V) which consists of the stochastic *CVa* and *CVi* building block models. Table 1 summarizes these distinctions.

Table 1
Different Views of the Conventional Model

Conventional Model as Developed by the BLS
1. Model does not focus on probabilistic concepts, rather is actuarial or demographic in nature.
2. Model produces only one primary result: worklife expectancy.
3. Model generates worklife for the entire population at all ages, called *CV,* actuarial, or demographic model – provides recipes.
4. Model generates worklife expectancies for the active population at peak and post-peak labor force participation ages, called *CVa* model. With additional assumptions, labor force participation for actives in pre-peak participation was verbally described.
5. Worklife expectancies for the inactive population are zero at peak and post-peak labor force participation ages, called *CVi* model.

Conventional Model Development New to This Paper, Called *LPd* Model
1. Model is formulated to be probabilistic in nature.
2. Model produces all of the characteristics of labor force activity: mean (worklife expectancy), median, mode, standard deviation, skewness, kurtosis, and probability intervals.
3. Model produces all of the characteristics of labor force activity for the active population (*CVa* model) , inactive population (*CVi* model) , and the entire population (*CV*) model.
4. Model produces all of the characteristics of labor force activity for all ages, whether pre or post-peak labor force participation ages.

Results Common to BLS Development and Skoog-Ciecka Development of the Conventional Model
1. Both models produce the same worklife expectancies for the entire population.
2. Both models produce the same worklife expectancies for the active population at peak and post-peak labor force participation ages.
3. Both models produce the same worklife expectancies for the inactive population at peak and post-peak labor force participation ages.
4. There are no other common results because the conventional model as developed by the BLS only focused on worklife expectancies.

Additional Comments
1. When *CV*, *CVa*, and *CVi* are used in the context of the original BLS development, these models are not probabilistic but are demographic/actuarial in nature.
2. When *CV*, *CVa,* and *CVi* are used in the context of the *LPd* model developed in this paper, these models are probabilistic but yield the same expectancies as the original BLS models, and many other results as well.

Neither the BLS nor Richards mistakenly allocated $Tw_x$ over only the $lp_x$ (i.e., actual actives, as opposed to the "closed population" of $l_x pp_{x_{peak}}$) actives in (5), for pre-peak ages, without acknowledgment of the need for such modifications at those pre-peak ages. Unfortunately the HPR estimator will be shown at (18) to do exactly this, but with the minor modification that the exact age $x$ quantity $lp_x$ is replaced by smoothed $Lp_x$ (HPR's $A_x$). In HPR's worked examples, the labor force is *not* closed, either under their Markov assumption or, after they switch the exposition to $pp_x$ terms involving data from the U.S. economy, under this alternative which they use in estimation.[9]

## IV. Probabilistic Interpretation of the *CVa* Part of the *LPd* Model at Peak and Post-Peak Participation Ages and the HPR, Nelson, and Smith Estimators for the Median

### A. Distribution Function and Survival Function at Peak and Post Peak Participation Ages for *CVa*

There is additional notation in *Bulletin 2135*, which we use to develop the probabilistic structure of the model *CVa*. $Q_x^s$ is defined there to be the rate of total labor force separations $S_x$ within the age interval $x$ to $x$+1:

$$Q_x^s = \frac{S_x}{Lw_x} = \frac{(Lw_x - Lw_{x+1})}{Lw_x} = 1 - \frac{Lw_{x+1}}{Lw_x},$$

which in our notation is

(6) $$Q_x^s = 1 - \frac{L_{x+1} pp_{x+1}}{L_x pp_x} = 1 - \frac{(l_{x+1} + l_{x+2})}{(l_x + l_{x+1})} \frac{pp_{x+1}}{pp_x}$$

With an end of period transition assumption, a person is credited with one additional year of activity or one year until final separation if separation occurs at the end of this interval. Recall that in this *CVa* model, all separations are final. One value of the *YA* and *YFS* probability mass function is established:

(7) $$p_{YFS}(x, a, 1; CVa) = Q_x^s$$

where $p_{YFS}(x, a, 1; CVa)$ is the probability that an active person age $x$ will finally separate from the labor force in one year in the *CVa* model.

We let $P_x^s$ indicate the probability that we do *not* separate at age $x$+1, so that

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[9]This HPR discussion is more consistent with a final alternative model we will develop in Section VI, the *LPi* model.

(8) $$P_x^s = 1 - Q_x^s = \frac{L_{x+1}}{L_x}\frac{pp_{x+1}}{pp_x}.$$

The same development leads to the general probability of finally separating after exactly *j* additional years

(9) $$p_{YFS}(x,a,j;CVa) = P_x^s P_{x+1}^s \dots P_{x+j-1}^s Q_{x+j}^s$$

and the probability of remaining active for more than *j* additional years is

(10) $$\sum_{i>j}^{111-x} p_{YFS}(x,a,i;CVa) = P_x^s P_{x+1}^s \dots P_{x+j}^s = 1 - \sum_{i=1}^{j} p_{YFS}(x,a,i;CVa);$$

the right hand side of (10) is the probability of remaining active through age *x+j*, so that at least *j*+1 more periods of activity are experienced.

We may then define, as usual, the *distribution function F(x,a,t)* for the years to final separation random variable as $YFS_{x,a}$,

(11) $$F(x,a,t) = \sum_{j=1}^{t} p_{YFS}(x,a,j;CVa) = P[YFS_{x,a} \le t],$$

The *survival function* associated with a random variable is traditionally denoted by *T*, and is defined as:

(12) $$T_{YFS_{x,a}}(t) = \mathrm{P}[YFS_{x,a} > t].$$

Evidently from (11) and (12), since $P[YFS_{x,a} \le t] + P[YFS_{x,a} > t] = 1$,

(13) $$T_{YFS_{x,a}}(t) = 1 - F(x,a,t),$$

so that (indicating the previous equalities used) we have

(14) $$P_x^s P_{x+1}^s \dots P_{x+t}^s \overset{(10),(11)}{=} 1 - F(x,a,t) \overset{(13)}{=} T_{YFS_{x,a}}(t).$$

giving us a closed form expression for the survivor function:

(15) $$T_{YFS_{x,a}}(t) = P_x^s P_{x+1}^s \dots P_{x+t}^s.$$

Since from (6) and (8),

$$P_{x+j}^s = \frac{L_{x+j+1}}{L_{x+j}}\frac{pp_{x+j+1}}{pp_{x+j}},$$

the telescoping nature of the survivor function (15) upon substitution produces a simplified expression for the survivor function of the *CVa* model:

$$T_{YFS_{x,a}}(t) = \frac{L_{x+t}}{L_x}\frac{pp_{x+t}}{pp_x}. \tag{16}$$

We will utilize this immediately below to find the appropriate estimators of the medians of *YFS* and *YA* in the *CVa* model, which we will recognize as the HPR estimating equation.

B. Probability Foundation for the YFS Estimator Used by HPR

Assume that there is an integer $k_x$ such that

$$T_{YFS_{x,a}}(k_x) = \frac{1}{2}. \tag{17}$$

Then by the construction of the survivor function, ½ of the population experiences 1,2,…up to and including $k_x$ additional years until final separation, and ½ receives more. $k_x$ is by definition the median of the distribution of the $YFS_{x,a}$ random variable. Solving the equation which sets the survivor function to ½ produces an appropriate estimator for the median time until final exit of *any* random variable. Substituting the $k_x$ definition (17) into the particular survivor expression (16) for this model, gives[10]

$$\frac{1}{2} = \frac{L_{x+k_x}pp_{x+k_x}}{L_x pp_x} \tag{18}$$

$$L_{x+k_x}pp_{x+k_x} = \frac{1}{2}L_x pp_x. \tag{18a}$$

We have shown that for the model *CVa*, starting at an age at or beyond the peak, half of the original $L_x pp_x$ active population gets $k_x$ years until final separation when $F(x,a,k_x) = ½$ exactly. But (18a), with consistent estimators of the population parameters $\{L_x\}$ and $\{pp_x\}$, is precisely the HPR computation or estimating equation. This provides that estimator with its first proper justification. For male actives, in the conventional model, and at or beyond peak participation, the HPR estimator is defined in the sample by an equality which is true in the population. Since this conventional model is "once out, always out" of the labor force, the random variables $YFS_{x,a}$ and $YA_{x,a}$ are here synonymous, and so the development above shows that $k_x$ provides an appropriate estimator for both the median of years to final labor force separation *and* the median years of future activity (not worklife expectancy), since these concepts are identical in this model. This is indeed ironic, because the

[10]In general, interpolation is required to determine both $k_x$ via (18) and the median of the empirical distribution function (11), but the result is the same as that proved above because the same interpolation is made in both equations.

median years to retirement that HPR claims to estimate generally is touted as being a "capacity" measure, as distinguished from the "years of activity" or worklife measure. In this one model, and in this limited domain, where the HPR estimator is justified, the two concepts of *YFS* and *YA* are the same, so its theoretical advantage of measuring the different concept of capacity has disappeared. We note that the HPR estimator has a probabilistic foundation for men at and beyond peak-participation ages; but it is outside this domain of justification when applied to pre-peak ages and to women, and possibly even to post-peak men unless still another regularity condition holds.

The expression (8), applied to $P^s_{x+j}$ shows that there may be logical problems with this model if $pp_{x+j+1} > pp_{x+j}$, since the "probability" in (8) could exceed 1. Equivalently, $Q^s_{x+j}$ would then be negative. We have seen that this occurs generally before the age of peak participation; it may also occur at later ages, as is shown in our Table 3, generated from the HPR (2001) participation rates at various ages. When the participation rate beyond the peak is not monotonically decreasing, and the monotonicity violation is not overcome in the product ratio in (8) by the force of mortality, such "negative probabilities" in the $YFS_{x,a}$ probabilities will occur. A sufficient condition to rule out "negative probabilities" and permit a bona fide probability distribution is the following monotonicity condition:

Monotonicity Condition (MC): $pp_{x+j+1} < pp_{x+j}$ for $x+j$ beyond $x_{peak}$

As we have noted, in the conventional model years to final separation and years of additional labor force activity are synonymous, so there is only one probability mass function: $p_{YA}\left(x,a,s;CVa\right) \equiv p_{YFS}\left(x,a,s;CVa\right)$. Further, we were fortunate to have been able to derive closed form expressions for these objects, a luxury not possible in an unrestricted Markov model or in other models of YFS and YA such as the *LPi* model below.

C. Probability Mass Function at Peak and Post-Peak Participation for *CVa*

From those closed form expressions, we may observe the following recursion:

(19)
$$p_{YFS}\left(x,a,t;CVa\right) = \left(1-Q^s_x\right) p_{YFS}\left(x+1,a,t-1;CVa\right) = P^s_x\, p_{YFS}(x+1,a,t-1;CVa), \quad t > 1,\, x < TA$$

and

(20)
$$p_{YFS}\left(x,a,t;CVa\right) = Q^s_x,\; t=1 \;\; x < TA\text{-}1,$$

where $Q_x^s$ gives the most immediate[11] separation probability at age *x* and *TA* is the terminal age.

Here (20) may be regarded as a boundary condition and (19) as the main recursion. From the end of period transition assumption and the definition of the *TA*, the terminal age (here 111), (20) is modified for *x=TA*-1 and *x=TA* to:

(21) $$p_{YFS}\left(TA\text{ -}1,a,1;CVa\right)=Q_{TA}^s=1\ ,$$

(22) $$p_{YFS}\left(TA,a,1;CVa\right)=0 \text{ and } p_{YFS}\left(TA,a,0;CVa\right)=1\ ,$$

which may be regarded as global conditions.

We now call the model given by (19)-(22) a partial version of the *LPd* model (developed in Section V) because it incorporates the mortality probability ("*L*") and participation probability ("$P$") of the *LPE* model, along with strong dependence ("*d*") assumptions regarding mortality (as usual) but now also participation. With $Q_x^s$ given by (6), we have "stochasticized" the conventional worklife model, at and beyond the age of peak participation and have specified a condition (MC) to ensure that the probability mass functions are well defined. Further, the recursive treatment of (19)-(22) indicates that the *LPd* model will satisfy the *Markov property*—future probabilities depend on no more past history than the current state, and in fact is a first-order Markov process.

Before extending the *LPd* model to earlier ages, we report on empirical work illustrating one development of the theory so far. Table 2 uses participation data from each of the two HPR papers. We replicate in columns (2) and (4) their YFS calculations for the total male population and male high school (only) graduates, using the demographic method described above, which makes no allowance for pre-peak entry. We then calculate, in adjacent columns (3) and (5), the median from what we have derived above as the probability mass function associated with the current development of the *LPd* model in (19)-(22). The computed results display the theoretical equivalences noted above, i.e., columns (3) and (5) yield the same results as columns (2) and (4), respectively.

Table 3 displays the associated probability mass functions at selected ages. The non-monotonicity in the HPR economy-wide participation rates causes the embarrassment of some negative "probabilities" at ages less than 67. Because of the placement of these violations, the estimate of the median is not affected. Were such a blip in the participation probability violating the MC to occur around the estimated median, multiple solutions for the median estimating equation could occur. The pmf derivation to this point is useful theoretically, providing an alternative derivation, and practically, demonstrating weaknesses in the model that had gone unnoticed. We note that our column (1) heading in Table 3 could alternatively been labeled "Years until Final Separation," and "Years of Activity" could have been in the title of Table 2 rather than "Years to Final Separation."

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[11]The choice of timing of the transition—here the end of the period transition (EPT)—makes this exactly one year. With a mid-period model, it would be ½ year, and with a beginning of period model it would be zero. This choice of timing matches up empirically best with the HPR model, but is quite inconsequential.

Table 2
Comparison of HPR Median Years to Final Separation for Men and Median of Years to Final Separation Calculated with Estimates from the "pmf" Generated by the *LPd* Model with End of Period Transitions

| Age<br>(1) | HPR's YFS Total *<br>(2) | Median *LPd* Model for All Men<br>(3) | HPR's YFS High School Diploma**<br>(4) | Median *LPd* Model High School Diploma<br>(5) |
|---|---|---|---|---|
| 18 | | 46.1 | 45.3 | 45.3 |
| 19 | | 44.1 | 43.6 | 43.6 |
| 20 | | 42.7 | 42.1 | 42.1 |
| 21 | | 41.3 | 40.9 | 40.9 |
| 22 | | 40.0 | 39.8 | 39.8 |
| 23 | | 38.8 | 38.7 | 38.7 |
| 24 | | 37.7 | 37.8 | 37.8 |
| 25 | 36.7 | 36.7 | 36.7 | 36.7 |
| 26 | 35.6 | 35.6 | 35.7 | 35.7 |
| 27 | 34.6 | 34.6 | 34.7 | 34.7 |
| 28 | 33.6 | 33.6 | 33.7 | 33.7 |
| 29 | 32.6 | 32.6 | 32.7 | 32.7 |
| 30 | 31.6 | 31.6 | 31.7 | 31.7 |
| 31 | 30.6 | 30.6 | 30.7 | 30.7 |
| 32 | 29.6 | 29.6 | 29.8 | 29.8 |
| 33 | 28.6 | 28.6 | 28.8 | 28.8 |
| 34 | 27.6 | 27.6 | 27.8 | 27.8 |
| 35 | 26.6 | 26.6 | 26.8 | 26.8 |
| 36 | 25.7 | 25.7 | 25.8 | 25.8 |
| 37 | 24.7 | 24.7 | 24.8 | 24.8 |
| 38 | 23.7 | 23.7 | 23.8 | 23.8 |
| 39 | 22.7 | 22.7 | 22.8 | 22.8 |
| 40 | 21.7 | 21.7 | 21.9 | 21.9 |
| 41 | 20.7 | 20.7 | 20.9 | 20.9 |
| 42 | 19.8 | 19.8 | 20.0 | 20.0 |
| 43 | 18.8 | 18.8 | 19.0 | 19.0 |
| 44 | 17.8 | 17.8 | 18.0 | 18.0 |
| 45 | 16.8 | 16.8 | 17.1 | 17.1 |
| 46 | 15.9 | 15.9 | 16.2 | 16.2 |
| 47 | 14.9 | 14.9 | 15.2 | 15.2 |
| 48 | 13.9 | 13.9 | 14.3 | 14.3 |
| 49 | 13.0 | 13.0 | 13.3 | 13.3 |
| 50 | 12.1 | 12.1 | 12.4 | 12.4 |
| 51 | 11.1 | 11.1 | 11.5 | 11.5 |
| 52 | 10.3 | 10.3 | 10.6 | 10.6 |
| 53 | 9.4 | 9.4 | 9.7 | 9.7 |
| 54 | 8.6 | 8.6 | 8.7 | 8.7 |
| 55 | 7.8 | 7.8 | 8.0 | 8.0 |
| 56 | 7.1 | 7.1 | 7.0 | 7.0 |
| 57 | 6.5 | 6.5 | 6.3 | 6.3 |
| 58 | 5.8 | 5.8 | 5.6 | 5.6 |
| 59 | 5.1 | 5.1 | 4.8 | 4.8 |
| 60 | 4.4 | 4.4 | 4.2 | 4.2 |
| 61 | 3.6 | 3.6 | 3.6 | 3.6 |
| 62 | 4.2 | 4.2 | 4.2 | 4.2 |
| 63 | 4.1 | 4.1 | 5.1 | 5.1 |
| 64 | 3.8 | 3.8 | 5.0 | 5.0 |
| 65 | 4.7 | 4.7 | 5.5 | 5.5 |
| 66 | 4.3 | 4.3 | 4.8 | 4.8 |
| 67 | 5.0 | 5.0 | 5.3 | 5.3 |
| 68 | 5.0 | 5.0 | 4.2 | 4.2 |

*Column 2 is from HPR, Appendix A, Table 3, *Journal of Forensic Economics*, 1997, 10 (2) and computed with (18).
**Column 4 is from HPR, Appendix A, Table 3, *Journal of Forensic Economics*, 2001, 14 (3), and computed with (18).

Table 3
"Probability Mass Functions" for Men with a High School Diploma at Various Ages
Generated by *LPd* Model with End of Period Transitions
Using the HPR Participation Rates*

| Years of Activity (1) | Age 20 (2) | Age 30 (3) | Age 40 (4) | Age 50 (5) | Age 60 (6) | Age 67 (7) | Age 68 (8) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -0.03873 | -0.00277 | 0.02104 | 0.01267 | 0.06013 | -0.01882 | 0.1901 |
| 2 | -0.02559 | 0.01015 | 0.00502 | 0.02666 | 0.17389 | 0.19368 | 0.09225 |
| 3 | -0.01842 | 0.00378 | 0.00308 | 0.01668 | 0.13226 | 0.09398 | 0.12527 |
| 4 | 0.01215 | -0.00041 | 0.00758 | 0.00472 | 0.11046 | 0.12763 | 0.05991 |
| 5 | -0.01605 | 0.00288 | 0.00569 | 0.04653 | 0.09482 | 0.06104 | 0.16143 |
| 6 | -0.00669 | 0.00829 | 0.01762 | 0.00459 | 0.02554 | 0.16447 | 0.37103 |
| 7 | -0.00201 | -0.0033 | 0.00936 | 0.0436 | 0.08473 | 0.37801 | |
| 8 | 0.00384 | 0.0032 | 0.00541 | 0.0501 | -0.00599 | | |
| 9 | -0.00429 | .000171 | 0.00048 | 0.03815 | 0.06162 | | |
| 10 | 0.01207 | 0.00352 | 0.02266 | 0.05611 | 0.0299 | | |
| 11 | -0.003 | 0.02051 | 0.01143 | 0.0421 | 0.04061 | | |
| 12 | 0.011 | 0.00489 | 0.02405 | 0.12176 | 0.01942 | | |
| 13 | 0.0041 | 0.003 | 0.01504 | 0.09261 | 0.05233 | | |
| 14 | -0.0004 | 0.00739 | 0.00425 | 0.07734 | 0.12027 | | |
| 15 | 0.00312 | 0.00554 | 0.04198 | 0.06639 | | | |
| 16 | 0.00898 | 0.01717 | 0.00414 | 0.01788 | | | |
| 17 | -0.00358 | 0.00912 | 0.03933 | 0.05933 | | | |
| 18 | 0.00346 | 0.00528 | 0.04519 | -0.00419 | | | |
| 19 | 1.92E-04 | .000468 | 0.03442 | 0.04315 | | | |
| 20 | 0.00382 | 0.02208 | 0.05062 | 0.02094 | | | |
| 21 | 0.02222 | 0.01114 | 0.03798 | 0.02843 | | | |
| 22 | 0.0053 | 0.02344 | 0.10983 | 0.0136 | | | |
| 23 | 0.00325 | 0.01466 | 0.08354 | 0.03664 | | | |
| 24 | 0.008 | 0.00415 | 0.06977 | 0.08421 | | | |
| 25 | 0.00601 | 0.0409 | 0.05989 | | | | |
| 26 | 0.01861 | 0.00403 | 0.01613 | | | | |
| 27 | 0.00989 | 0.03833 | 0.05352 | | | | |
| 28 | 0.00572 | 0.04404 | -0.00378 | | | | |
| 29 | .000051 | 0.03354 | 0.03892 | | | | |
| 30 | 0.02393 | 0.04932 | 0.01889 | | | | |
| 31 | 0.01207 | 0.03701 | 0.02565 | | | | |
| 32 | 0.0254 | 0.10703 | 0.01227 | | | | |
| 33 | 0.01589 | 0.08141 | 0.03305 | | | | |
| 34 | 0.00449 | 0.06799 | 0.07596 | | | | |
| 35 | 0.04433 | 0.05836 | | | | | |
| 36 | 0.00437 | 0.01572 | | | | | |
| 37 | 0.04153 | 0.05215 | | | | | |
| 38 | 0.04773 | -0.00368 | | | | | |
| 39 | 0.03634 | 0.03793 | | | | | |
| 40 | 0.05345 | 0.0184 | | | | | |
| 41 | 0.04011 | 0.02499 | | | | | |

(Continued)

Table 3 (continued)
"Probability Mass Functions" for Men with a High School Diploma at Various Ages Generated by *LPd* Model with End of Period Transitions Using the HPR Participation Rates*

| Years of Activity (1) | Age 20 (2) | Age 30 (3) | Age 40 (4) | Age 50 (5) | Age 60 (6) | Age 67 (7) | Age 68 (8) |
|---|---|---|---|---|---|---|---|
| 42 | 0.11599 | 0.01195 | | | | | |
| 43 | 0.08822 | 0.03221 | | | | | |
| 44 | 0.07368 | 0.07402 | | | | | |
| 45 | 0.06325 | | | | | | |
| 46 | 0.01703 | | | | | | |
| 47 | 0.05652 | | | | | | |
| 48 | -0.00399 | | | | | | |
| 49 | 0.0411 | | | | | | |
| 50 | 0.01994 | | | | | | |
| 51 | 0.02709 | | | | | | |
| 52 | 0.01295 | | | | | | |
| 53 | 0.0349 | | | | | | |
| 54 | 0.08022 | | | | | | |
| Total | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

*Participation Rates from HPR, Appendix A, Table 1, *Journal of Forensic Economics*, 2001, 14 (3).

Table 3 shows entire mass functions for selected ages from which the medians in column (5) of Table 2 were computed. For example, the median is 4.2 for 68-year-old males in Table 2, column (5). The last column in Table 3 is the corresponding pmf from the *LPd* model with end of period transitions. Counting probability from "left to right," the accumulated mass is .46753 at 4 years; and interpolation gives 4.2 = 4.0 + [(.5 – .46753)/.16141]. Again, we are using end-of-period transitions. Had midpoint transitions been used and probability mass distributed uniformly over one-year intervals, as we have done in other work (SC, 2003), the interpolated median would be 4.2 years as well.

D. Nelson Median and the Smith and HPR Medians from Open Populations

Frasca and Winger (1989), using 1979-80 transition probabilities, estimated the mean age at final separation from the labor force and years to final separation. Their approach was innovative: starting active at a particular age, they calculated (per the Markov model) the probability of being active at some future age (their age *n*), and they multiplied that probability by the probability of staying inactive thereafter. The foregoing product of probabilities gave the probability of separating at age *n*, which was multiplied by *n* and summed over all ages to yield the expected value of age at final separation. Frasca and Winger compared their means to unpublished Shirley Smith estimates of the Nelson median, but they also criticized Nelson's median because it did not properly account for temporary inactives from an initial active population and temporary actives who were initially inactive.

Frasca and Winger (1989) offer the following definitions:

$l_j$ = the number of currently active workers at base age *j*.

$f_{jk}$ = the number of currently active workers at base age *j* who finally separate from the labor force at or before (*sic*; this needs correction, below) age *k*.

$g_{jk}$ = the number of currently active workers at base age *j* who are temporarily inactive at age *k*

$h_{jk}$ = the number of currently inactive workers at base age *j* who are active at age *k*.

Letting *M* be the median age at final separation and *N* be the Nelson median age, the two defining equations are asserted (by Frasca and Winger) to be:

(23) $$\sum_{k=j+1}^{k=M} \frac{f_{jk}}{l_j} = \frac{1}{2}$$

and

(24) $$\sum_{k=j+1}^{k=N} \frac{(f_{jk} + g_{jk} - h_{jk})}{l_j} = \frac{1}{2}$$

We offer here some comment on this loose notation, which generally pervades much of this demographic literature and may escape the casual reader. It is understood that the terms $f_{jk}, g_{jk}$ and $h_{jk}$ in this equation are counts of persons with the stated attributes drawn from a reasonably large sample of size $l_j$. The division of these terms by $l_j$, passage to the limit, and the strong law of large numbers imply that these are equivalently statements about underlying probabilities. In the first (sampling) interpretation, (23) and (24) implicitly define *M* and *N* as estimators. The latter (population) interpretation shows that the same equation results from the probability limits of these estimators. The argument is that since equation (24) contains two terms different from equation (23), the two estimators *M* and *N* differ. *M* clearly converges to the population median, and the estimator of the Nelson median must therefore be inconsistent. Perhaps because the additional terms $(+g_{jk})$ and $(-h_{jk})$ are of opposite sign, Frasca and Winger asserted that "It is unlikely, however, that errors in aggregation will cause the Nelson median to significantly depart from the true median." Empirically we find more than 3 years of difference for men younger than age 57 and women under age 51 (see column (6) in Tables 4 and 5) and can sign the gap with a function of the transition probabilities.

While the conclusion that the two medians are unequal in general is correct, the Frasca and Winger reasoning is unfortunately unclear. First, the definition of $f_{jk}$ must be changed, to read:[12]

$f_{jk}$ = the number of currently active workers at base age $j$ who finally separate from the labor force *exactly* at age $k$.

In the Frasca-Winger definition of the Nelson median, the definition of $h_{jk}$ is puzzling since their previous discussion is entirely about the currently active labor force, $l_j$, (there were no inactives in $l_j$ by its own definition) and there can be *no* such inactives, *i.e.*, $h_{jk} = 0$ for all $k$. This makes the expression confusing, but, in our analysis below, technically correct. Readers don't expect authors to add terms which are zero, however, and therefore will not be sure of their understanding the development beyond this point; they may even look for other interpretations, one of which we offer below, requiring re-writing more definitions. Now, for the actives or participants to drop to ½ the previous level, we can either count those active after $N$ or those who have become inactive at $N$: the cup is both half full and half empty. The latter is what (24) incorrectly attempts. Those inactive at age $N$ will include two mutually exclusive groups: those who have become permanently inactive ("finally separated") at any of the ages $j+1, \ldots, N$ and those who are temporarily inactive at $N$ (*i.e.* will resume additional activity later). The first group is $\sum_{k=j+1}^{k=N} f_{jk}$ and the latter group is $g_{jN}$. Thus the proper equation in the Frasca-Winger notation defining the Nelson median age, for a closed population of $l_j$ actives and 0 inactives, the age when ½ of the original actives are inactive, is:

$$\sum_{k=j+1}^{k=N} f_{jk} + g_{jN} = \frac{1}{2} l_j \tag{25}$$

Now $g_{jN}$ in the sample [equivalently, $\operatorname*{plim}_{l_j \to \infty} (g_{jN} / l_j)$] is generally greater than 0, since there will always be people who are temporarily inactive at ages $j+s$ from the $l_j$ group, provided ${}^a p^i_{j+s} > 0$ for some $s > 0$ and ${}^i p^a_{j+s+t} > 0$ for some $t$=1,2,…, as is true empirically before the truncation age when inactivity has become an absorbing state. This positive number of people in $g_{jN}$ establishes the general result that, within the Markov model, the Nelson median estimator will be less than the consistent median estimator $M$ for large $l_j$, for a population which initially started active, since a positive term allows

[12]Equivalently, the summation could be dropped, and the original definition employed. We shall make the first indicated correction, however, assuming that the authors intended their summations. Both definitions assume that the medians occur at an integer age; this does not happen in practice, but simplifies the exposition, so we adopt this expository device.

the upper limit in the sum to be reached earlier. The empirical results in the Tables 4 and 5 demonstrate this, and we record this as the following theorem.

Nelson Median Underestimation Theorem

In the Markov model, for ages before inactivity becomes an absorbing state, for those starting active in a closed population, the Nelson "median" estimator *N* will in all finite samples and in the limit be less than the consistent estimator *M* of the population median defined by (23), with the corrected $f_{jk}$ definition.

In fact, consider the

$$l_j\{\sum_{s_1,s_2,\ldots s_{N-1}} {}^{a}p_j^{s_1}\ {}^{s_1}p_{j+1}^{s_2}\ldots {}^{s_{1N-1}}p_{N-1}^{i}\}\text{ inactives at }N.$$

This group will be equal to $\sum_{k=j+1}^{k=N} f_{jk}$ precisely when there is no escape from inactivity to activity beyond *N*, *i.e.* when ${}^{i}p_{N-1+s}^{a} = 0$ for $s = 1,2,\ldots$. This happens identically (*i.e.* for all ages *j* at which a Nelson median might be computed) when and only when inactivity and death together form an absorbing state, hence the ${}^{i}p_j^{a} = 0$ all *j* requirement for consistency. This is the key assumption of the conventional model of worklife as discussed above and establishes, with a different proof, the result:

Nelson Median Corollary

In the conventional model of worklife, where only one transition to inactivity beyond the peak age of labor force participation is allowed, the Nelson median will consistently estimate the population median.

To be clear, our use of the term Nelson median thus far assumes that one starts with a Current Population Survey type sample, fits transition probabilities for a Markov model, chooses the desired age *j* and starts with a radix of $l_j$ actives and 0 inactives. Using the Markov model to determine the number of actives at each subsequent age, one determines the Nelson median age *N* as the first age when the number of actives hits $l_j/2$. The results above deal with this usage for "Nelson median." Since Nelson did not publish his transition probabilities, and we have not seen any work that has replicated his results, this appears to be a fair interpretation, and is certainly consistent with Frasca-Winger.

We have been assuming a *closed* population, but one might imagine the effects of a misdirected application to a set of actives which was *not* a part of a closed population. The first such estimator we consider we call the Smith median age, following widely cited but unpublished calculations of Shirley Smith (undated). We may analyze this situation as we did in our earlier theorem. Since the previously puzzling $h_{jk}$ flows are now nonzero, we need more notation. We will trace through the Markov model flows into the active and inactive states commencing with ${}^{a}l_j$ actives and ${}^{i}l_j$ inactives (now both are non-zero). At any future age older than *j*, the initial actives will appear somewhere in the first row and the initial inactives somewhere in the second row of the following table.

| Status at *j* | Active at *k* | Temporarily Inactive at *k* | Finally Separated at *k* |
|---|---|---|---|
| Active at *j* | $m_{jk}$ | $g_{jk}$ | $f_{j,j+1},\ldots, f_{jk}$ |
| Inactive at *j* | $h_{jk}$ | $n_{jk}$ | $p_{j,j+1},\ldots, p_{jk}$ |

While our Nelson median earlier was derived by equating the inactives to ½ $l_j$ , evidently the equation ${}^{a}l_j = m_{jk} + g_{jk} + \sum_{s=j+1}^{s=k} f_{js}$ accounts for the ${}^{a}l_j$ for any age *k*>*j*, and so is an identity. If half are inactive at *N*, the Nelson median, the previous equation for the Nelson median (25), $.5\,{}^{a}l_j = g_{jN} + \sum_{s=j+1}^{s=N} f_{js}$ , may be subtracted from the identity to conclude that $.5\,{}^{a}l_j = m_{jN}$ , a result we use in the next paragraph. This expresses the Nelson median in terms of the actives, but does not yield the previous interesting inequality.

Now the estimating equation for *S*, the Smith median age of final labor force separation, in the previous notation is $.5\,{}^{a}l_j = m_{jS} + h_{jS}$ , since the two terms on the right hand side give the actives at age *S*. Equating the left hand sides, $m_{jS} + h_{jS} = m_{jN}$ links the Smith and Nelson medians. Since $m_{jk}$ is monotonically decreasing in *k* (for *k* in the relevant range—people transition out of activity except at very young ages), and $h_{jk}$ is positive (before very old ages—there are some transitions from inactivity to activity, except again in the *CVa* model), *S* must exceed *N*. This inequality holds in finite samples, and upon division by the radices, in the probability limit. We thus have a second inequality involving the flavors of Nelson-Smith median estimation:

Nelson-Smith Median Inequality

The Smith median exceeds the Nelson median. The result is intuitive: starting from the same number of actives, if there are inactives to flow into active status, it takes longer for the actives to drop to ½ their previous level.

We see this result in our columns (4) and (5) of Table 4 and Table 5.

Table 4
Comparison of Skoog-Ciecka *YFS* Medians for All Men with Nelson, Smith and HPR Medians

| Age (1) | SC Mean (2) | SC Median (3) | Nelson (4) | Smith (5) | Col. 3 – 4 (6) | Col. 3 – 5 (7) | HPR 2001 (8) | Col. 4 – 8 (9) | Col. 5 – 8 (10) |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 47.29 | 48.40 | 45.00 | 50.18 | 3.41 | -1.78 | | | |
| 17 | 46.34 | 47.41 | 44.01 | 48.79 | 3.41 | -1.38 | | | |
| 18 | 45.40 | 46.43 | 43.02 | 46.92 | 3.41 | -0.49 | 46.6 | -3.58 | 0.32 |
| 19 | 44.46 | 45.44 | 42.03 | 45.18 | 3.41 | 0.26 | 44.6 | -2.57 | 0.58 |
| 20 | 43.53 | 44.46 | 41.04 | 43.73 | 3.41 | 0.73 | 43.0 | -1.96 | 0.73 |
| 21 | 42.59 | 43.47 | 40.06 | 42.38 | 3.41 | 1.09 | 41.7 | -1.64 | 0.68 |
| 22 | 41.66 | 42.49 | 39.07 | 40.98 | 3.41 | 1.51 | 40.4 | -1.33 | 0.58 |
| 23 | 40.73 | 41.50 | 38.09 | 39.63 | 3.42 | 1.87 | 39.1 | -1.01 | 0.53 |
| 24 | 39.80 | 40.52 | 37.11 | 38.31 | 3.42 | 2.21 | 38.0 | -0.89 | 0.31 |
| 25 | 38.86 | 39.54 | 36.12 | 37.04 | 3.42 | 2.50 | 36.9 | -0.78 | 0.14 |
| 26 | 37.93 | 38.56 | 35.14 | 35.81 | 3.42 | 2.75 | 35.8 | -0.66 | 0.01 |
| 27 | 36.99 | 37.57 | 34.15 | 34.64 | 3.42 | 2.93 | 34.8 | -0.65 | -0.16 |
| 28 | 36.05 | 36.59 | 33.17 | 33.51 | 3.42 | 3.08 | 33.8 | -0.63 | -0.29 |
| 29 | 35.12 | 35.61 | 32.18 | 32.42 | 3.43 | 3.19 | 32.8 | -0.62 | -0.38 |
| 30 | 34.18 | 34.63 | 31.20 | 31.37 | 3.43 | 3.26 | 31.8 | -0.60 | -0.43 |
| 31 | 33.25 | 33.65 | 30.22 | 30.35 | 3.43 | 3.30 | 30.8 | -0.58 | -0.45 |
| 32 | 32.33 | 32.67 | 29.24 | 29.34 | 3.43 | 3.33 | 29.8 | -0.56 | -0.46 |
| 33 | 31.40 | 31.70 | 28.26 | 28.36 | 3.43 | 3.34 | 28.8 | -0.54 | -0.44 |
| 34 | 30.48 | 30.72 | 27.29 | 27.38 | 3.44 | 3.34 | 27.8 | -0.51 | -0.42 |
| 35 | 29.56 | 29.75 | 26.31 | 26.41 | 3.44 | 3.34 | 26.8 | -0.49 | -0.39 |
| 36 | 28.64 | 28.77 | 25.34 | 25.44 | 3.44 | 3.33 | 25.9 | -0.56 | -0.46 |
| 37 | 27.72 | 27.80 | 24.36 | 24.47 | 3.44 | 3.33 | 24.9 | -0.54 | -0.43 |
| 38 | 26.81 | 26.83 | 23.39 | 23.51 | 3.44 | 3.32 | 23.9 | -0.51 | -0.39 |
| 39 | 25.89 | 25.86 | 22.42 | 22.55 | 3.45 | 3.31 | 22.9 | -0.48 | -0.35 |
| 40 | 24.98 | 24.90 | 21.45 | 21.60 | 3.45 | 3.30 | 21.9 | -0.45 | -0.30 |
| 41 | 24.07 | 23.93 | 20.48 | 20.67 | 3.45 | 3.26 | 20.9 | -0.42 | -0.23 |
| 42 | 23.17 | 22.97 | 19.52 | 19.73 | 3.45 | 3.24 | 20.0 | -0.48 | -0.27 |
| 43 | 22.26 | 22.01 | 18.56 | 18.80 | 3.45 | 3.21 | 19.0 | -0.44 | -0.20 |
| 44 | 21.36 | 21.05 | 17.60 | 17.86 | 3.45 | 3.19 | 18.0 | -0.40 | -0.14 |
| 45 | 20.47 | 20.10 | 16.65 | 16.93 | 3.45 | 3.17 | 17.1 | -0.45 | -0.17 |
| 46 | 19.57 | 19.14 | 15.70 | 16.00 | 3.44 | 3.14 | 16.1 | -0.40 | -0.10 |
| 47 | 18.68 | 18.19 | 14.76 | 15.09 | 3.44 | 3.10 | 15.2 | -0.44 | -0.11 |
| 48 | 17.80 | 17.25 | 13.82 | 14.19 | 3.43 | 3.06 | 14.3 | -0.48 | -0.11 |
| 49 | 16.92 | 16.31 | 12.89 | 13.29 | 3.42 | 3.02 | 13.3 | -0.41 | -0.01 |
| 50 | 16.05 | 15.37 | 11.97 | 12.41 | 3.40 | 2.96 | 12.4 | -0.43 | 0.01 |
| 51 | 15.19 | 14.44 | 11.06 | 11.55 | 3.38 | 2.89 | 11.5 | -0.44 | 0.05 |
| 52 | 14.34 | 13.52 | 10.18 | 10.70 | 3.35 | 2.82 | 10.6 | -0.42 | 0.10 |
| 53 | 13.50 | 12.61 | 9.31 | 9.88 | 3.30 | 2.73 | 9.8 | -0.49 | 0.08 |
| 54 | 12.68 | 11.71 | 8.47 | 9.09 | 3.24 | 2.62 | 8.9 | -0.43 | 0.19 |
| 55 | 11.88 | 10.83 | 7.66 | 8.34 | 3.17 | 2.49 | 8.1 | -0.44 | 0.24 |
| 56 | 11.09 | 9.96 | 6.88 | 7.63 | 3.08 | 2.33 | 7.4 | -0.52 | 0.23 |
| 57 | 10.33 | 9.12 | 6.14 | 6.96 | 2.97 | 2.16 | 6.7 | -0.56 | 0.26 |
| 58 | 9.59 | 8.30 | 5.47 | 6.40 | 2.84 | 1.90 | 6.1 | -0.63 | 0.30 |
| 59 | 8.88 | 7.53 | 4.85 | 5.90 | 2.68 | 1.63 | 5.4 | -0.55 | 0.50 |
| 60 | 8.22 | 6.80 | 4.32 | 5.52 | 2.47 | 1.28 | 4.8 | -0.48 | 0.72 |
| 61 | 7.60 | 6.13 | 3.88 | 5.25 | 2.25 | 0.88 | 4.0 | -0.12 | 1.25 |
| 62 | 7.04 | 5.54 | 3.54 | 5.09 | 2.00 | 0.45 | 4.7 | -1.16 | 0.39 |
| 63 | 6.52 | 4.99 | 3.25 | 5.02 | 1.75 | -0.03 | 5.2 | -1.95 | -0.18 |
| 64 | 6.05 | 4.54 | 3.00 | 4.97 | 1.54 | -0.43 | 4.9 | -1.90 | 0.07 |

(Continued)

Table 4 (continued)
Comparison of Skoog-Ciecka *YFS* Medians for All Men with Nelson, Smith and HPR Medians

| Age (1) | SC Mean (2) | SC Median (3) | Nelson (4) | Smith (5) | Col. 3 – 4 (6) | Col. 3 – 5 (7) | HPR 2001 (8) | Col. 4 – 8 (9) | Col. 5 – 8 (10) |
|---|---|---|---|---|---|---|---|---|---|
| 65 | 5.61 | 4.11 | 2.84 | 4.97 | 1.27 | -0.86 | 5.2 | -2.36 | -0.23 |
| 66 | 5.22 | 3.76 | 2.69 | 4.97 | 1.07 | -1.21 | 5.4 | -2.71 | -0.43 |
| 67 | 4.85 | 3.47 | 2.55 | 4.89 | 0.91 | -1.42 | 5.2 | -2.65 | -0.31 |
| 68 | 4.52 | 3.20 | 2.44 | 4.72 | 0.76 | -1.52 | 4.9 | -2.46 | -0.18 |
| 69 | 4.22 | 2.99 | 2.37 | 4.51 | 0.62 | -1.52 | 5.0 | -2.63 | -0.49 |
| 70 | 3.94 | 2.82 | 2.30 | 4.26 | 0.52 | -1.44 | 4.7 | -2.40 | -0.44 |
| 71 | 3.67 | 2.62 | 2.19 | 4.03 | 0.43 | -1.41 | | | |
| 72 | 3.40 | 2.41 | 2.02 | 3.70 | 0.40 | -1.29 | | | |
| 73 | 3.13 | 2.22 | 1.90 | 3.32 | 0.32 | -1.10 | | | |
| 74 | 2.90 | 2.11 | 1.86 | 2.87 | 0.25 | -0.76 | | | |
| 75 | 2.66 | 1.94 | 1.76 | 2.49 | 0.19 | -0.55 | | | |

Columns 2, 3, 4, & 5 computed with SC transition probabilities. See SC (2003) for Columns 2 & 3. Column 8 provided by HPR in private correspondence with SC and calculated by SC using the HPR procedure and using pmf's for the *LPd* model. Here "median" refers to median years to labor force separation, not age at separation.

Table 5
Comparison of Skoog-Ciecka *YFS* Medians for All Women with Nelson, Smith and HPR Medians

| Age (1) | SC Mean (2) | SC Median (3) | Nelson (4) | Smith (5) | Col. 3 – 4 (6) | Col. 3 – 5 (7) | HPR 2001 (8) | Col. 4 – 8 (9) | Col. 5 – 8 (10) |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 46.01 | 46.53 | 42.83 | 48.77 | 3.70 | -2.24 | | | |
| 17 | 45.03 | 45.53 | 41.83 | 47.03 | 3.70 | -1.50 | | | |
| 18 | 44.05 | 44.54 | 40.84 | 45.22 | 3.70 | -0.68 | 45.0 | -4.16 | 0.22 |
| 19 | 43.08 | 43.54 | 39.85 | 43.53 | 3.69 | 0.01 | 43.3 | -3.45 | 0.23 |
| 20 | 42.10 | 42.55 | 38.85 | 42.24 | 3.69 | 0.31 | 42.0 | -3.15 | 0.24 |
| 21 | 41.12 | 41.55 | 37.86 | 41.13 | 3.69 | 0.42 | 40.9 | -3.04 | 0.23 |
| 22 | 40.14 | 40.56 | 36.87 | 39.95 | 3.69 | 0.61 | 39.6 | -2.73 | 0.35 |
| 23 | 39.16 | 39.56 | 35.87 | 38.79 | 3.69 | 0.77 | 38.4 | -2.53 | 0.39 |
| 24 | 38.18 | 38.57 | 34.88 | 37.65 | 3.68 | 0.92 | 37.4 | -2.52 | 0.25 |
| 25 | 37.21 | 37.57 | 33.89 | 36.52 | 3.68 | 1.05 | 36.4 | -2.51 | 0.12 |
| 26 | 36.23 | 36.58 | 32.90 | 35.40 | 3.68 | 1.18 | 35.3 | -2.40 | 0.10 |
| 27 | 35.25 | 35.58 | 31.90 | 34.31 | 3.68 | 1.27 | 34.4 | -2.50 | -0.09 |
| 28 | 34.28 | 34.59 | 30.91 | 33.25 | 3.68 | 1.34 | 33.3 | -2.39 | -0.05 |
| 29 | 33.30 | 33.60 | 29.92 | 32.21 | 3.67 | 1.39 | 32.4 | -2.48 | -0.19 |
| 30 | 32.33 | 32.60 | 28.93 | 31.19 | 3.67 | 1.41 | 31.5 | -2.57 | -0.31 |
| 31 | 31.36 | 31.61 | 27.94 | 30.16 | 3.67 | 1.45 | 30.5 | -2.56 | -0.34 |
| 32 | 30.39 | 30.62 | 26.96 | 29.13 | 3.66 | 1.49 | 29.5 | -2.54 | -0.37 |
| 33 | 29.42 | 29.63 | 25.97 | 28.12 | 3.66 | 1.51 | 28.5 | -2.53 | -0.38 |
| 34 | 28.45 | 28.64 | 24.98 | 27.13 | 3.65 | 1.51 | 27.5 | -2.52 | -0.37 |
| 35 | 27.49 | 27.65 | 24.00 | 26.14 | 3.65 | 1.51 | 26.6 | -2.60 | -0.46 |
| 36 | 26.53 | 26.66 | 23.01 | 25.12 | 3.64 | 1.54 | 25.5 | -2.49 | -0.38 |
| 37 | 25.57 | 25.67 | 22.03 | 24.10 | 3.64 | 1.57 | 24.5 | -2.47 | -0.40 |
| 38 | 24.61 | 24.68 | 21.05 | 23.07 | 3.63 | 1.61 | 23.5 | -2.45 | -0.43 |
| 39 | 23.66 | 23.70 | 20.08 | 22.05 | 3.62 | 1.65 | 22.4 | -2.32 | -0.35 |
| 40 | 22.71 | 22.72 | 19.11 | 21.04 | 3.61 | 1.68 | 21.4 | -2.29 | -0.36 |
| 41 | 21.77 | 21.73 | 18.14 | 20.04 | 3.59 | 1.69 | 20.3 | -2.16 | -0.26 |
| 42 | 20.83 | 20.75 | 17.18 | 19.03 | 3.57 | 1.72 | 19.3 | -2.12 | -0.27 |
| 43 | 19.90 | 19.78 | 16.23 | 18.02 | 3.55 | 1.76 | 18.3 | -2.07 | -0.28 |
| 44 | 18.98 | 18.80 | 15.28 | 17.02 | 3.52 | 1.78 | 17.3 | -2.02 | -0.28 |
| 45 | 18.07 | 17.83 | 14.35 | 16.05 | 3.48 | 1.78 | 16.3 | -1.95 | -0.25 |

(Continued)

Table 5 (continued)
Comparison of Skoog-Ciecka *YFS* Medians for All Women with Nelson, Smith and HPR Medians

| Age (1) | SC Mean (2) | SC Median (3) | Nelson (4) | Smith (5) | Col. 3 – 4 (6) | Col. 3 – 5 (7) | HPR 2001 (8) | Col. 4 – 8 (9) | Col. 5 – 8 (10) |
|---|---|---|---|---|---|---|---|---|---|
| 46 | 17.16 | 16.87 | 13.44 | 15.10 | 3.43 | 1.77 | 15.3 | -1.86 | -0.20 |
| 47 | 16.27 | 15.91 | 12.54 | 14.18 | 3.37 | 1.73 | 14.3 | -1.76 | -0.12 |
| 48 | 15.39 | 14.96 | 11.66 | 13.28 | 3.29 | 1.68 | 13.4 | -1.74 | -0.12 |
| 49 | 14.53 | 14.01 | 10.81 | 12.39 | 3.20 | 1.62 | 12.5 | -1.69 | -0.11 |
| 50 | 13.69 | 13.08 | 9.99 | 11.53 | 3.09 | 1.55 | 11.6 | -1.61 | -0.07 |
| 51 | 12.86 | 12.17 | 9.19 | 10.70 | 2.97 | 1.47 | 10.7 | -1.51 | 0.00 |
| 52 | 12.05 | 11.27 | 8.43 | 9.90 | 2.84 | 1.37 | 9.9 | -1.47 | 0.00 |
| 53 | 11.26 | 10.38 | 7.70 | 9.15 | 2.69 | 1.23 | 9.0 | -1.30 | 0.15 |
| 54 | 10.50 | 9.53 | 7.01 | 8.45 | 2.52 | 1.08 | 8.3 | -1.29 | 0.15 |
| 55 | 9.77 | 8.70 | 6.37 | 7.79 | 2.34 | 0.91 | 7.6 | -1.23 | 0.19 |
| 56 | 9.07 | 7.91 | 5.76 | 7.15 | 2.15 | 0.76 | 6.8 | -1.04 | 0.35 |
| 57 | 8.40 | 7.15 | 5.18 | 6.55 | 1.97 | 0.60 | 6.0 | -0.82 | 0.55 |
| 58 | 7.75 | 6.44 | 4.66 | 5.99 | 1.78 | 0.45 | 5.5 | -0.84 | 0.49 |
| 59 | 7.13 | 5.77 | 4.18 | 5.51 | 1.59 | 0.26 | 5.1 | -0.92 | 0.41 |
| 60 | 6.56 | 5.16 | 3.76 | 5.08 | 1.40 | 0.08 | 4.6 | -0.84 | 0.48 |
| 61 | 6.03 | 4.64 | 3.41 | 4.75 | 1.22 | -0.11 | 4.0 | -0.59 | 0.75 |
| 62 | 5.57 | 4.18 | 3.12 | 4.48 | 1.07 | -0.30 | 4.3 | -1.18 | 0.18 |
| 63 | 5.14 | 3.79 | 2.89 | 4.25 | 0.90 | -0.46 | 4.1 | -1.21 | 0.15 |
| 64 | 4.76 | 3.45 | 2.70 | 4.08 | 0.75 | -0.63 | 3.9 | -1.20 | 0.18 |
| 65 | 4.42 | 3.14 | 2.53 | 4.01 | 0.61 | -0.87 | 4.6 | -2.07 | -0.59 |
| 66 | 4.10 | 2.88 | 2.35 | 4.11 | 0.52 | -1.23 | 4.2 | -1.85 | -0.09 |
| 67 | 3.82 | 2.69 | 2.22 | 4.33 | 0.47 | -1.64 | 4.9 | -2.68 | -0.57 |
| 68 | 3.58 | 2.57 | 2.16 | 4.43 | 0.41 | -1.86 | 5.0 | -2.84 | -0.57 |
| 69 | 3.37 | 2.47 | 2.13 | 4.34 | 0.34 | -1.87 | 4.3 | -2.17 | 0.04 |
| 70 | 3.15 | 2.35 | 2.08 | 4.07 | 0.27 | -1.72 | 4.2 | -2.12 | -0.13 |
| 71 | 2.91 | 2.17 | 1.97 | 3.73 | 0.21 | -1.56 | | | |
| 72 | 2.67 | 1.96 | 1.83 | 3.36 | 0.13 | -1.4 | | | |
| 73 | 2.43 | 1.80 | 1.69 | 2.98 | 0.11 | -1.18 | | | |
| 74 | 2.20 | 1.68 | 1.59 | 2.60 | 0.09 | -0.92 | | | |
| 75 | 1.99 | 1.53 | 1.47 | 2.19 | 0.06 | -0.66 | | | |

Columns 2, 3, 4, and 5 computed with SC transition probabilities. See SC (2003) for Columns 2 and 3. Column 8 calculated by SC using the HPR procedure and using pmf's for the *LPd* model. Here "median" refers to median years to labor force separation, not age at separation.

Consider at age 16 the number of inactives to be 71,421 and the number of actives to be 25,360—these are the data for all U.S males in 1970 found in Table A-1, page 35, columns (12) and (13) of *Bulletin 2135*. We consider these published columns, which depict the beginning of age $x$ stationary state inactives and actives, ${}^{i}l_x$ and ${}^{a}l_x$, but unpublished work of Smith whose estimates we have been able to replicate, show that she employed this kind of mixed or stationary distribution of actives and inactives. Her work used concepts corresponding to columns (21) and (22), ${}^{\bullet}L_x^i$ and ${}^{\bullet}L_x^a$ of *Bulletin 2135*—the upper left dot indicates the mixed radix or stationary state. Looking down column (22) for the 16-year-olds, half of the ${}^{\bullet}L_{16}^a$ (analogous to the $A_{16}$ in HPR's terminology) of 30,149 is reached between ages 69 and 70. Starting at age 25, $A_{25}$ =72,852, which drops to half between ages 64 and 65. This "Smith median years to separation" for 16-year-olds is thus 53+, while the "Smith median years to separation" for 25-year-olds is 39+. It is obviously absurd to imagine that aging nine years results in ones' median years to final separation decreasing 14 years; but this follows the application of this methodology involving open populations to the Markov model. Similarly, HPR's methodology, with *no* articulated model, to the extent that their economy-wide data were similar to Smith's implied participation rates, suffers from these same problems. HPR simply multiply a sequence of participation rates by a life-table induced survival function to determine (model-free) actives. In addition to running into the problems of the above example, by eschewing a model, their results become uninterpretible—"measurement without theory" in economic parlance. The fundamental issues are theoretical and methodological. Estimators may be evaluated because the model in which they reside permits it; recipes cannot be evaluated. It is irrelevant that the HPR estimator gives answers which appear reasonable; so too does a naïve estimator announcing "median years to separation" as equal to $62.6 - x$ for age $x$ (62.5 or 62.6 appear to be the average ages people who are eligible have begun taking Social Security payments).

The last paragraph suggests that the HPR estimator, appealing to economy-wide participation rates, may be an approximation to Smith's $pp_x$ weighted Markov estimator, to the extent that the Markov estimator's implied $pp_x$ participations match those of the economy; this matching is a minor issue we have discussed elsewhere. Our point here is that Smith at least ran her algorithm through a (Markov) model. We have seen that her mixed radix median estimator in turn was an over-approximation to Nelson's pure radix median estimator, which itself was shown to be inconsistent within the general Markov model. Putting these inequalities together, there is no logical inequality in finite samples or in probability limits between our consistent estimates and Smith's (as shown in column (7) of Tables 4 and 5) and HPR's (as we will show in Table 6, columns 4 and 7).

It is not surprising that such unflattering conclusions for the HPR estimator would appear. The object being estimated, the median years until final separation, requires first a model which specifies the probability of each and every sample path occurring. When years in the labor force was being investigated, different models of individual behavior which implied the same economy-wide worklife (observationally equivalent models) could be tolerated; with years to final separation, such underidentification is intolerable—the object under consideration is a statement about characteristics of sample paths. Within a well specified model, the median is required to be estimated. If the solution is to nominate an *ad hoc* estimating equation without reference to any model, the predictable result will be that such an estimator will have poor statistical properties for almost all rigorously specified models. There may be some model (other than our *CVa*) which rationalizes the *ad hoc* HPR estimation procedure, but there is no guarantee that, if such a model is found, it will be any more desirable or even reasonable. It is because the world has moved beyond the *CVa* model that we find the HPR estimates intrinsically unsuitable.

Table 6
Comparison of SC and HPR Median Years to Final Separation

| | | All Men | | | All Women | |
|---|---|---|---|---|---|---|
| Age (1) | SC (2) | HPR (3) | Col. 2 –3 (4) | SC (5) | HPR (6) | Col.5 – 6 (7) |
| 18 | 46.4 | 46.6 | -0.2 | 44.5 | 45.0 | -0.5 |
| 19 | 45.4 | 44.6 | 0.8 | 43.5 | 43.3 | 0.2 |
| 20 | 44.5 | 43.0 | 1.5 | 42.6 | 42.0 | 0.6 |
| 21 | 43.5 | 41.7 | 1.8 | 41.6 | 40.9 | 0.7 |
| 22 | 42.5 | 40.4 | 2.1 | 40.6 | 39.6 | 1.0 |
| 23 | 41.5 | 39.1 | 2.4 | 39.6 | 38.4 | 1.2 |
| 24 | 40.5 | 38.0 | 2.5 | 38.6 | 37.4 | 1.2 |
| 25 | 39.5 | 36.9 | 2.6 | 37.6 | 36.4 | 1.2 |
| 26 | 38.6 | 35.8 | 2.8 | 36.6 | 35.3 | 1.3 |
| 27 | 37.6 | 34.8 | 2.8 | 35.6 | 34.4 | 1.2 |
| 28 | 36.6 | 33.8 | 2.8 | 34.6 | 33.3 | 1.3 |
| 29 | 35.6 | 32.8 | 2.8 | 33.6 | 32.4 | 1.2 |
| 30 | 34.6 | 31.8 | 2.8 | 32.6 | 31.5 | 1.1 |
| 31 | 33.7 | 30.8 | 2.9 | 31.6 | 30.5 | 1.1 |
| 32 | 32.7 | 29.8 | 2.9 | 30.6 | 29.5 | 1.1 |
| 33 | 31.7 | 28.8 | 2.9 | 29.6 | 28.5 | 1.1 |
| 34 | 30.7 | 27.8 | 2.9 | 28.6 | 27.5 | 1.1 |
| 35 | 29.8 | 26.8 | 3.0 | 27.7 | 26.6 | 1.1 |
| 36 | 28.8 | 25.9 | 2.9 | 26.7 | 25.5 | 1.2 |
| 37 | 27.8 | 24.9 | 2.9 | 25.7 | 24.5 | 1.2 |
| 38 | 26.8 | 23.9 | 2.9 | 24.7 | 23.5 | 1.2 |
| 39 | 25.9 | 22.9 | 3.0 | 23.7 | 22.4 | 1.3 |
| 40 | 24.9 | 21.9 | 3.0 | 22.7 | 21.4 | 1.3 |
| 41 | 23.9 | 20.9 | 3.0 | 21.7 | 20.3 | 1.4 |
| 42 | 23.0 | 20.0 | 3.0 | 20.8 | 19.3 | 1.5 |
| 43 | 22.0 | 19.0 | 3.0 | 19.8 | 18.3 | 1.5 |
| 44 | 21.1 | 18.0 | 3.1 | 18.8 | 17.3 | 1.5 |
| 45 | 20.1 | 17.1 | 3.0 | 17.8 | 16.3 | 1.5 |
| 46 | 19.1 | 16.1 | 3.0 | 16.9 | 15.3 | 1.6 |
| 47 | 18.2 | 15.2 | 3.0 | 15.9 | 14.3 | 1.6 |
| 48 | 17.3 | 14.3 | 3.0 | 15.0 | 13.4 | 1.6 |
| 49 | 16.3 | 13.3 | 3.0 | 14.0 | 12.5 | 1.5 |
| 50 | 15.4 | 12.4 | 3.0 | 13.1 | 11.6 | 1.5 |
| 51 | 14.4 | 11.5 | 2.9 | 12.2 | 10.7 | 1.5 |
| 52 | 13.5 | 10.6 | 2.9 | 11.3 | 9.9 | 1.4 |
| 53 | 12.6 | 9.8 | 2.8 | 10.4 | 9.0 | 1.4 |
| 54 | 11.7 | 8.9 | 2.8 | 9.5 | 8.3 | 1.2 |
| 55 | 10.8 | 8.1 | 2.7 | 8.7 | 7.6 | 1.1 |
| 56 | 10.0 | 7.4 | 2.6 | 7.9 | 6.8 | 1.1 |
| 57 | 9.1 | 6.7 | 2.4 | 7.2 | 6.0 | 1.2 |
| 58 | 8.3 | 6.1 | 2.2 | 6.4 | 5.5 | 0.9 |
| 59 | 7.5 | 5.4 | 2.1 | 5.8 | 5.1 | 0.7 |
| 60 | 6.8 | 4.8 | 2.0 | 5.2 | 4.6 | 0.6 |
| 61 | 6.1 | 4.0 | 2.1 | 4.6 | 4.0 | 0.6 |
| 62 | 5.5 | 4.7 | 0.8 | 4.2 | 4.3 | -0.1 |
| 63 | 5.0 | 5.2 | -0.2 | 3.8 | 4.1 | -0.3 |
| 64 | 4.5 | 4.9 | -0.4 | 3.5 | 3.9 | -0.4 |
| 65 | 4.1 | 5.2 | -1.1 | 3.1 | 4.6 | -1.5 |
| 66 | 3.8 | 5.4 | -1.6 | 2.9 | 4.2 | -1.3 |
| 67 | 3.5 | 5.2 | -1.7 | 2.7 | 4.9 | -2.2 |
| 68 | 3.2 | 4.9 | -1.7 | 2.6 | 5.0 | -2.4 |
| 69 | 3.0 | 5.0 | -2.0 | 2.5 | 4.3 | -1.8 |
| 70 | 2.8 | 4.7 | -1.9 | 2.4 | 4.2 | -2.0 |

See Table 4 for Columns 2 and 3; see Table 5 for Columns 5 and 6.

Indeed, our econometric view of the HPR estimates is that, since the *CVa* model they implicitly estimate is nested within the general Markov model, they really provide specification error tests—Hausman (1978). To the extent that they differ from our unrestricted estimates, the data are inconsistent with the *CVa* model. The tables below elaborate upon our main point, that the differences between SC's empirical results and HPR's are theoretical and not sampling issues. The theoretical inequalities we have proved do not establish an inequality either way, and this non-result is borne out in Table 6.

In Table 4, columns (2) and (3), we report, for all education categories combined, our mean (for reference only) and then our consistent estimate of the true median years to final separation for men (See Table 5 for all women.). We then run through the Markov model the Nelson and Smith type median calculations. The estimator in column (4) was the one we attributed to Nelson and begins with a closed population of 100,000 actives at each age; it calculates when 50,000 remain active. It is analogous to the calculation of worklife within the Markov model, and, as Smith pointed out, required re-setting the radix at each age, to force any inactives at later ages to have come from a closed population. We then performed the Smith median calculations, again within the Markov model, in column (5). It is column (6), our consistent median calculations less the Nelson pure radix calculations, which our first underestimation theorem speaks to, *i.e.*, columns (3)-(4), which show a difference of more than three years until ages in the 50s. Our theory establishing that the Smith estimator will exceed the Nelson estimator is apparent, since column (5) exceeds column (4). Column (8) is the HPR calculation of YFS. Based on a snapshot of the overall U.S. economy at a point in time, 1998-1999, we would expect that the HPR column (8) would mimic the Smith median in column (5) although it is based on 1997-1998 data. This is indeed the case—they usually are within a half of a year of each other. Thus, our differences with HPR are not "in the data" but "in the methodology," as we have maintained.

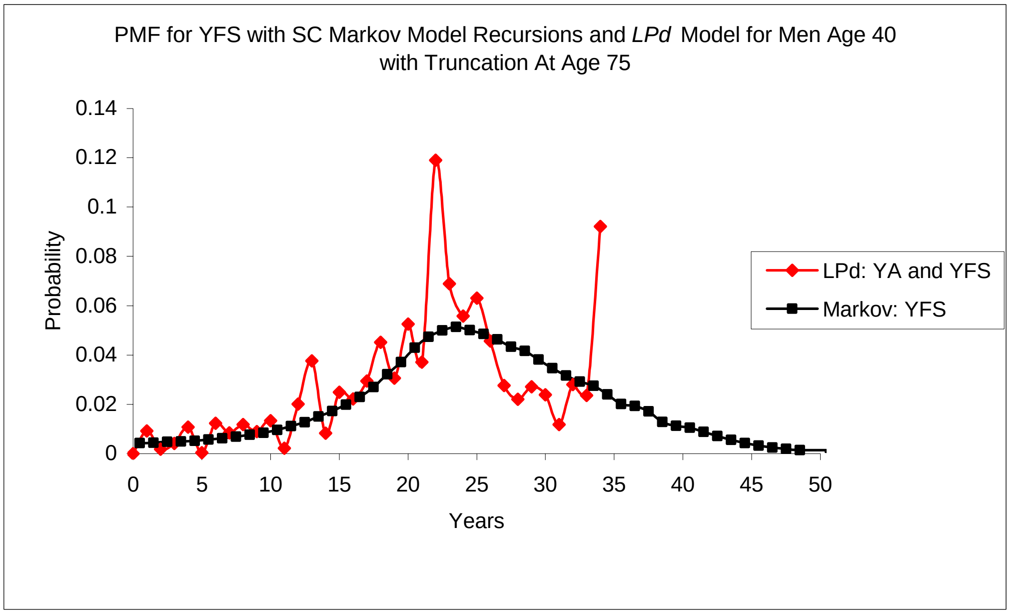
In Table 6 we present a direct comparison between SC medians for years to final separation and the YFS from HPR for all men and all women.[13] Several factors are important to understand this comparison. (1) There is a year of difference in the data, and it is not obvious how the years to final separation would be affected by whatever data differences may arise. (2) HPR ignore transition probabilities and use only population participation probabilities, while we do not estimate population participation rates directly because there is no need. (3) HPR estimates, by the results above, are inconsistent within the Markov model, a model sufficiently general that it includes all other worklife models as nested special cases. (4) The initial condition of the first inequality theorem requires a closed population. This condition is most nearly met for men between ages 25–55 when participation rates are approximately 90% and higher. The population is not completely closed and, using the terminology

[13]HPR do not present YFS for all men and all women in their most current paper, but rather for various educational groups. The YFS in Table 6 were supplied by HPR for men and were substantially replicated by SC; YFS for women were calculated by SC using the HPR procedure and using pmf's for the *LPd* model.

above for exposition, the term $h_{jk}$ (the number of currently inactive workers at base age *j* who are active at age *k*) plays a role in increasing the HPR median. At older ages (63 and beyond) when the inactive population is large and its contribution to future activity is inappropriately attributed to those who where initially active, the HPR estimator then overestimates the true median. In fact, the effect is so strong at later ages as to cause the HPR median to increase after it reaches a minimum at age 61 for men and age 64 for women. (5) For the Markov model, there is a well defined probability mass function for years of future activity. Recursions for this are presented in SC (2003), and an expression for them is implicit in Frasca and Winger. The SC estimates are consistent for the median of this probability mass function. With no such probability mass function or probability model defined by HPR, further comment on the properties of their estimator is difficult—what is it supposed to be estimating? We know it is inconsistent for the general Markov model; is there another model other than *CVa* for which their estimator is appropriate?

We illustrate two probability mass functions in Figure 2 for 40-year-old initially active men: the SC pmf for *YFS* which generates a median of 24.9 years as shown in Table 6 and the *LPd* pmf (which does double duty by serving as the pmf for both years of activity and years until final separation) with a median of 21.9 years—exactly the same as HPR's median. The upper panel of Figure 2 shows a final probability spike representing the combined probability for *YFS* at ages 75 and above, since participation rate data end at age 75. This spike does not affect the median. The median is the same in the lower panel where the pmf was generated by letting the final observed participation rate decline by 10% in each year for 15 additional years.

Of course, HPR do not use a pmf in their calculations, but Figure 2 shows the saw-tooth type pmf that provides the only known underlying probability foundation for their estimator. The jagged appearance of the *LPd* pmf, while troublesome, is less of a problem than the once-out, always-out nature of the conventional model, since this assumption leads to overestimating the probability of early final withdrawals and underestimating the probability of later final exits. This tendency may be observed in this figure. Assuming fulfillment of the monotonicity condition, the hallmarks of the conventional model at post-peak ages are (1) all activity at a future age is supplied by those active at the current age and (2) inactivity is an absorbing state which, for model purposes, is exactly like the death state. These assumptions are not generally true, but they affect median years to final separation in opposite directions. We argue below that the first assumption leads to overestimates and the second to underestimates. The net effect is an underestimate of the median for most ages before approximately age 62 (except for very young ages) and an overestimate for ages beyond 62. Figure 2, and similarly in Figure 3 but for women age 50, shows the *LPd* pmf function producing an underestimate of years to final separation because too much probability is "stacked" on small years until final separation.



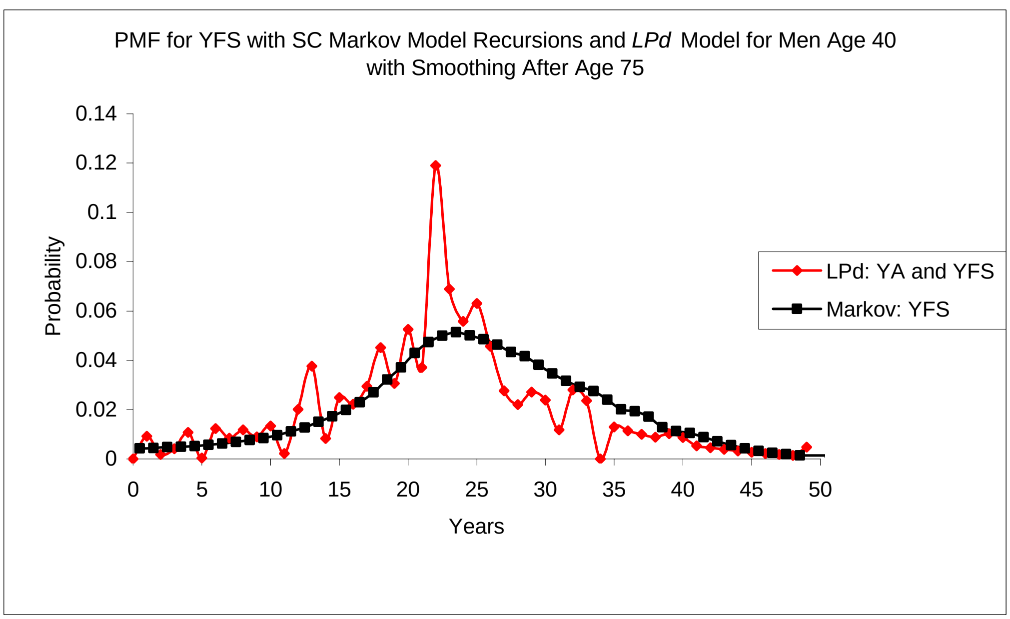
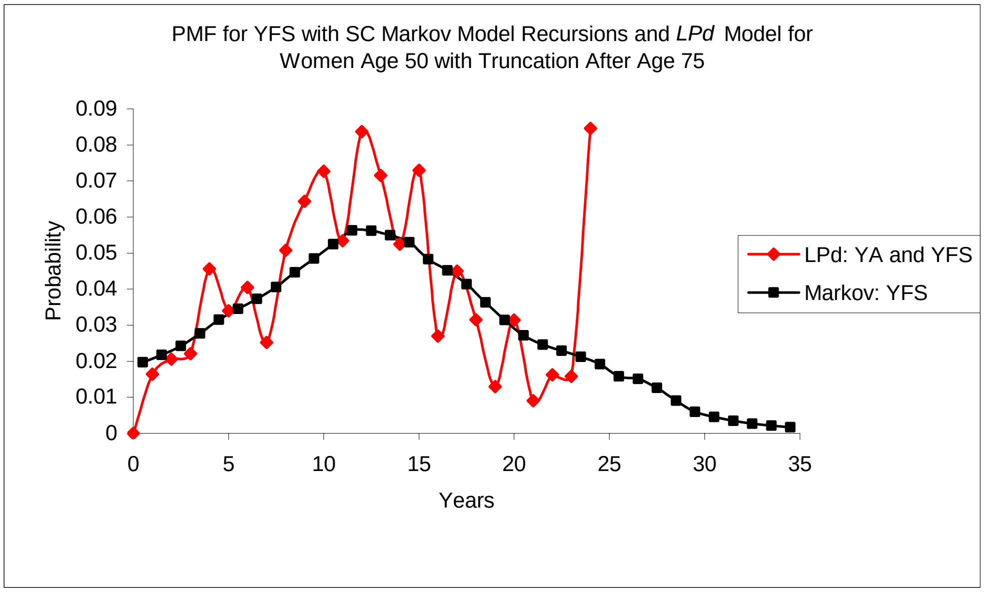




Figure 2
Probability Mass Functions for Initially Active Men Age 40: *YFS* Calculated with SC Recursions and Transition Probabilities, *LPd* Model Calculated with HPR 1998-99 Participation Rates



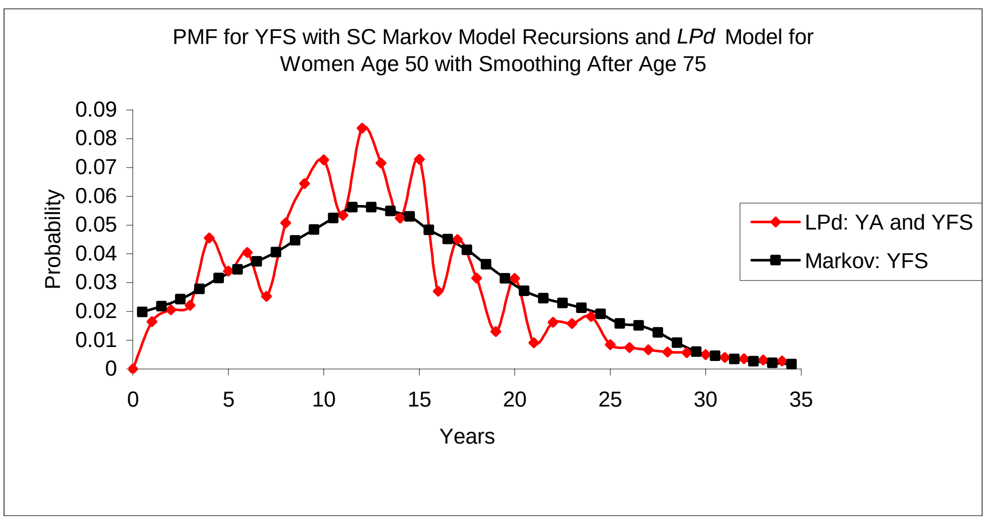




Figure 3
Probability Mass Functions for Initially Active Women Age 50: *YFS* Calculated with SC Recursions and Transition Probabilities, *LPd* Model Calculated with HPR 1998-99 Participation Rates

We illustrate the argument in the previous paragraph by considering transition probabilities at a few specific ages. The conventional model starting active, and therefore HPR's work on medians, implicitly (*via* 28 below) acts as if it overestimates ${}^{a}p_{x}^{a}$ and underestimates ${}^{i}p_{x}^{a}$—the former problem contributing to overestimated medians and the latter to underestimates. For middle-age men, the impact of overestimated ${}^{a}p_{x}^{a}$ is at its smallest since most males are active and remain active for several years. In this age group, underestimated ${}^{i}p_{x}^{a}$ is the main problem that causes HPR medians to be too small. For example, for males age 40, SC use ${}^{a}p_{40}^{a} = .975$; and HPR report $pp_{40} = .928$ and $pp_{41} = .922$ (HPR, 2001). The mortality adjusted ratio of participation rates (*i.e.*, the active-to-active transition probability that is implicit in the conventional model at post-peak participation ages and in HPR's work) is thus $(L_{41}/L_{40})(pp_{41}/pp_{40}) = (.997)(.922/.928) = .991$, using $(L_{41}/L_{40}) = .997$ as HPR do. Indeed, this implicit active-to-active transition probability is too high, but it cannot be too high by very much because ${}^{a}p_{40}^{a} = .975$ itself is so close to one. The main problem with the conventional model and HPR is not overestimated ${}^{a}p_{40}^{a}$ but rather underestimated ${}^{i}p_{40}^{a}$. The once-out-always-out nature of the conventional model, implicit in HPR's work, assumes that ${}^{i}p_{40}^{a} = 0$, but SC using the increment-decrement model have ${}^{i}p_{40}^{a} = .243$. This latter non-zero transition probability, and its feedback to activity through the main SC recursions which define the pmf, result in more years until final labor force separation as illustrated in Figure 2. This type of underestimate of the implicit inactive-to-active transition probabilities simply dominates empirically the small overestimates of active-to-active transition probabilities. At later ages the relative importance of the conventional model and HPR errors reverse. Take age 67 as an example. SC use ${}^{a}p_{67}^{a} = .759$. HPR have $pp_{67} = .275$, $pp_{68} = .255$, and a survival probability of $(L_{68}/L_{67}) = .975$; the implicit active-to-active transition probability is $(L_{68}/L_{67})(pp_{68}/pp_{67}) = (.975)(.255/.275) = .904$. The conventional model starting active and the implicit HPR inactive-to-active transition probability once again is zero, but that is not too far from ${}^{i}p_{65}^{a} = .052$ used by SC. Here, the conventional model and the HPR assumption of zero are not too far from the actual ${}^{i}p_{65}^{a}$ which is positive but close to zero. However, the impact of the *CVa*'s and HPR's large overestimate of ${}^{a}p_{67}^{a}$ dominates the smaller underestimate of ${}^{i}p_{67}^{a}$. This type of problem leads to the HPR estimate of the median years to final separation being 1.7 years too large as shown in Table 6.

## V. The Full *LPd* Model: Probabilistic Interpretation of the *CVa* and *CVi* Models at All Ages

We next describe a treatment of the conventional model which extends the *LPd* approach. The key observation is that the BLS approach to dealing with pre-peak ages may be modeled as a special case of the general Markov model, with strong restrictions on the transition probabilities. In particular, for *any pre-peak age x*, we define transition probabilities for a Markov process by

(26) $$ {}^{a}p_x^{a} = \frac{L_{x+1}}{L_x}\,,\ {}^{a}p_x^{i} = 0\ ,\ {}^{a}p_x^{d} = 1 - \frac{L_{x+1}}{L_x}\ , $$

and

(27) $$ {}^{i}p_x^{i} = \frac{L_{x+1}}{L_x}\left[1 - \frac{(pp_{x+1} - pp_x)}{(1 - pp_x)}\right],\ {}^{i}p_x^{a} = \frac{L_{x+1}}{L_x}\frac{(pp_{x+1} - pp_x)}{(1 - pp_x)},\ \text{and}\ {}^{i}p_x^{d} = 1 - \frac{L_{x+1}}{L_x}\,. $$

The equations in (26) contain probabilities from the active state and say that exits for actives occur only due to death, never to inactivity, in the pre-peak years. The numerator in the first term in (27) reverses the sign of the labor force separations, $Lp_x - Lp_{x+1}$, which occur in the later years; pre-peak, these are net accessions. Motivation for (27) is equation (10)[14] in *Bulletin 2135*.

Thus we have been able to capture, with a restricted Markov model, the BLS treatment of the pre-peak years; based on this description, one might no longer wish to call these "fictitious years" as the BLS did.

At and beyond the peak participation year, we continue with the specification of Markov transition probabilities. They follow as in the previously defined *CVa* and *CVi* models:

(28) $$ {}^{a}p_x^{a} = 1 - Q_x^{s} = \frac{L_{x+1}}{L_x}\frac{pp_{x+1}}{pp_x}\,,\ {}^{a}p_x^{i} = \frac{L_{x+1}}{L_x}\left(1 - \frac{pp_{x+1}}{pp_x}\right),\ {}^{a}p_x^{d} = 1 - \frac{L_{x+1}}{L_x} $$

(29) $$ {}^{i}p_x^{i} = \frac{L_{x+1}}{L_x}\,,\ {}^{i}p_x^{a} = 0\ ,\ {}^{i}p_x^{d} = 1 - \frac{L_{x+1}}{L_x}\,. $$

We call this extended model the full *LPd* model.[15] Its probabilities are consistent with the conventional model starting active for all ages, and it captures the strong state dependence of that model, even into the pre-peak years. Its worklife expectancies replicate those of the conventional model starting active, *CVa*, as shown in Table 7 by comparing columns (2) and (7). Its weighted average column (4) agrees with the demographic or actuarial version of the conventional model for all ages, shown in columns (5) and (6). Finally, because it is representable as a Markov model with restrictions, the machinery of the SC papers is available with which to completely analyze medians, modes, variances, standard errors, and probability intervals for this formalization or extension of the conventional model. The idea of the conventional model satisfying the Markov property for years before age $x_{peak}$ is new, as are non-zero worklife expectancies when commencing *inactive* for the pre-peak years—an extension of the conventional model to *CVi,* which the BLS did not treat theoretically or estimate. Probability mass functions for the conventional models (*CVa, CVi*) are new. Tables 7 and 8 are interpreted on page 83 and 84.

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[14]For reasons not understood by us, deaths are added back in by the BLS, a practice not followed here. At these early ages, in any event, this difference is inconsequential.

[15]These equations were chosen for consistency with both *Bulletin 2135* and HPR. The specification in the usual Markov model contains terms which are close but not identical. For example, ${}^{a}p_x^{d}$ and ${}^{i}p_x^{d}$ appear here as $(L_x - L_{x+1})/L_x$, whereas the probability of death at exact age $x$ is $(l_x - l_{x+1})/l_x$. We will not hesitate to use this exact age modification in our future work involving the *LPd* model.

Table 7
Comparison of Richards's Worklife Expectancies for Men with a High School Diploma with SC Mean Years of Activity Calculated with pmf Generated by theFull *LPd* Model with End of Period Transitions

| Age<br>(1) | SC Mean (WLE) for Initial Actives *LPd* Model<br>(2) | SC Mean (WLE) for Initial Inactives *LPd* Model<br>(3) | Weighted Average for Initial Actives and Inactives<br>(4) | WLE Demography Model<br>(5) | WLE Richards*<br>(6) | WLE for Initial Actives BLS Method<br>(7) |
|---|---|---|---|---|---|---|
| 18 | 42.5 | 31.8 | 39.1 | 39.1 | 39.1 | 42.4 |
| 19 | 41.5 | 28.0 | 38.4 | 38.4 | 38.5 | 41.5 |
| 20 | 40.6 | 25.0 | 37.7 | 37.7 | 37.8 | 40.6 |
| 21 | 39.6 | 20.2 | 37.0 | 36.9 | 37.0 | 39.6 |
| 22 | 38.7 | 14.5 | 36.2 | 36.1 | 36.2 | 38.7 |
| 23 | 37.8 | 11.0 | 35.3 | 35.3 | 35.4 | 37.7 |
| 24 | 36.8 | 7.5 | 34.5 | 34.4 | 34.5 | 36.8 |
| 25 | 35.9 | 4.0 | 33.6 | 33.6 | 33.6 | 35.8 |
| 26 | 34.9 | 2.4 | 32.7 | 32.7 | 32.8 | 34.9 |
| 27 | 34.0 | 0.0 | 31.8 | 31.8 | 31.9 | 34.0 |
| 28 | 33.1 | 0.0 | 30.9 | 30.9 | 31.0 | 33.0 |
| 29 | 32.1 | 0.0 | 30.1 | 30.0 | 30.1 | 32.1 |
| 30 | 31.3 | 0.0 | 29.2 | 29.2 | 29.2 | 31.3 |
| 31 | 30.4 | 0.0 | 28.3 | 28.3 | 28.3 | 30.3 |
| 32 | 29.5 | 0.0 | 27.4 | 27.4 | 27.5 | 29.5 |
| 33 | 28.6 | 0.0 | 26.5 | 26.5 | 26.6 | 28.5 |
| 34 | 27.7 | 0.0 | 25.7 | 25.6 | 25.7 | 27.7 |
| 35 | 26.8 | 0.0 | 24.8 | 24.8 | 24.8 | 26.8 |
| 36 | 25.9 | 0.0 | 23.9 | 23.9 | 24.0 | 25.9 |
| 37 | 25.0 | 0.0 | 23.1 | 23.0 | 23.1 | 25.0 |
| 38 | 24.1 | 0.0 | 22.2 | 22.2 | 22.3 | 24.0 |
| 39 | 23.2 | 0.0 | 21.3 | 21.3 | 21.4 | 23.1 |
| 40 | 22.3 | 0.0 | 20.5 | 20.4 | 20.5 | 22.3 |
| 41 | 21.4 | 0.0 | 19.6 | 19.6 | 19.7 | 21.4 |
| 42 | 20.6 | 0.0 | 18.8 | 18.7 | 18.8 | 20.6 |
| 43 | 19.8 | 0.0 | 17.9 | 17.9 | 18.0 | 19.7 |
| 44 | 19.0 | 0.0 | 17.1 | 17.0 | 17.2 | 19.0 |
| 45 | 18.2 | 0.0 | 16.2 | 16.2 | 16.3 | 18.1 |
| 46 | 17.3 | 0.0 | 15.4 | 15.4 | 15.5 | 17.3 |
| 47 | 16.4 | 0.0 | 14.6 | 14.6 | 14.7 | 16.4 |
| 48 | 15.6 | 0.0 | 13.8 | 13.7 | 13.9 | 15.6 |
| 49 | 14.7 | 0.0 | 12.9 | 12.9 | 13.0 | 14.7 |
| 50 | 13.9 | 0.0 | 12.1 | 12.1 | 12.2 | 13.9 |
| 51 | 13.1 | 0.0 | 11.3 | 11.3 | 11.4 | 13.0 |
| 52 | 12.3 | 0.0 | 10.5 | 10.5 | 10.6 | 12.3 |
| 53 | 11.5 | 0.0 | 9.7 | 9.7 | 9.8 | 11.5 |
| 54 | 10.8 | 0.0 | 9.0 | 8.9 | 9.1 | 10.7 |
| 55 | 10.0 | 0.0 | 8.2 | 8.2 | 8.3 | 10.0 |
| 56 | 9.4 | 0.0 | 7.5 | 7.4 | 7.5 | 9.3 |
| 57 | 8.7 | 0.0 | 6.7 | 6.7 | 6.8 | 8.6 |
| 58 | 8.1 | 0.0 | 6.0 | 6.0 | 6.1 | 8.0 |
| 59 | 7.4 | 0.0 | 5.3 | 5.3 | 5.4 | 7.4 |
| 60 | 7.0 | 0.0 | 4.7 | 4.7 | 4.8 | 7.0 |
| 61 | 6.7 | 0.0 | 4.1 | 4.1 | 4.2 | 6.7 |
| 62 | 6.6 | 0.0 | 3.5 | 3.5 | 3.6 | 6.6 |
| 63 | 6.7 | 0.0 | 3.1 | 3.0 | 3.1 | 6.7 |
| 64 | 6.9 | 0.0 | 2.7 | 2.6 | 2.7 | 6.9 |
| 65 | 7.0 | 0.0 | 2.3 | 2.3 | 2.4 | 7.0 |
| 66 | 7.1 | 0.0 | 2.0 | 2.0 | 2.1 | 7.1 |
| 67 | 7.0 | 0.0 | 1.8 | 1.8 | 1.9 | 7.0 |
| 68 | 6.7 | 0.0 | 1.6 | 1.6 | 1.7 | 6.7 |
| 69 | 6.6 | 0.0 | 1.4 | 1.4 | 1.5 | 6.6 |

(Continued)

Table 7 (continued)
Comparison of Richards's Worklife Expectancies for Men with a High School Diploma with SC Mean Years of Activity Calculated with pmf Generated by theFull *LPd* Model with End of Period Transitions

| Age (1) | SC Mean (WLE) for Initial Actives *LPd* Model (2) | SC Mean (WLE) for Initial Inactives *LPd* Model (3) | Weighted Average for Initial Actives and Inactives (4) | WLE Demography Model (5) | WLE Richards* (6) | WLE for Initial Actives BLS Method (7) |
|---|---|---|---|---|---|---|
| 70 | 6.4 | 0.0 | 1.2 | 1.2 | 1.3 | 6.3 |
| 71 | 6.1 | 0.0 | 1.1 | 1.1 | 1.1 | 6.1 |
| 72 | 5.9 | 0.0 | 0.9 | 0.9 | 1.0 | 5.9 |
| 73 | 5.6 | 0.0 | 0.8 | 0.8 | 0.9 | 5.6 |

*Richards and Abele, *Life and Worklife Expectancies*, Lawyers and Judges Publishing Co., 1999, Table 6. Columns 2, 3, 5, and 7 calculated with Richards' participation rates.

The pmf for the full *LPd* model with end-of-period transitions is given by (30a) – (30f):

Probability Mass Function for the *LPd* Model for Both Years of Activity and Years until Final Labor Force Separation with End-of-Period Transitions

(30a) $p_{YFS}\left(TA,a,0;LPd\right)=1$ and $p_{YFS}\left(TA,i,0;LPd\right)=1$

(30b) $p_{YFS}\left(x,a,0;LPd\right)=0,\ x\leq TA\text{-}1,$

(30c) $p_{YFS}\left(x,a,1;LPd\right)=Q_x^s,\ x\leq TA\text{-}1,$

(30d) $p_{YFS}\left(x,i,0;LPd\right)={}^ip_x^i\,p_{YFS}(x+1,i,0;LPd)+{}^ip_x^d,\ x\leq TA\text{-}1,$

(30e) $p_{YFS}\left(x,a,t;LPd\right)={}^ap_x^a\,p_{YFS}(x+1,a,t-1;LPd),\ x<TA$ and $t>1,$

(30f) $p_{YFS}\left(x,i,t;LPd\right)={}^ip_x^a\,p_{YFS}(x+1,a,t;LPd)+{}^ip_x^i\,p_{YFS}(x+1,i,t;LPd),$
$x<TA$ and $t\geq 1$

where (30a) is a global condition, (30b)—(30d) are boundary conditions, and (30e) – (30f) are the main recursions. Since YA and YFS are the same concept in this model, the recursions could have been subscripted with *YA* (years of activity random variable) instead. These recursions give, of course, the previous closed form solutions for what we have called *CVa* and *CVi*.

From the definition of ${}^ap_x^a$ in the post-peak years, it is evident that the monotonocity condition (MC) must be present here also, in order for such mass functions to be well defined. As a theoretical matter, employing "expectancies" or "medians" when it would not be possible to find a probability function of which these "expectancies" were the mathematical expectations or the "medians" were population medians would be strongly objectionable. We note that the MC held in the BLS *Bulletin 2135* data and in the Richards data we have used for exposition; it is was mildly violated in the HPR examples, and negative probabilities were even reported in Nelson (1983). The failure of MC represents an additional reason to reject the conventional model starting active, and estimators like HPR's which are predicated upon it.

Table 8
Probability Mass Functions for Men at Age 20 and 40 Generated by the Full *LPd* Model with End of Period Transitions Using Richards Participation Rates*

| Years of Activity (1) | Initially Active Age 20 (2) | Initially Inactive Age 20 (3) | Initially Active Age 40 (4) | Initially Inactive Age 40 (5) |
|---|---|---|---|---|
| 0 | 0 | 0.34732 | 0 | 1 |
| 1 | 0.00146 | 0.00103 | 0.00661 | 0 |
| 2 | 0.00152 | 0.00106 | 0.00812 | 0 |
| 3 | 0.00155 | 0.00114 | 0.00923 | 0 |
| 4 | 0.00154 | 1.19E-03 | 0.01087 | 0 |
| 5 | 0.00152 | 0.00137 | 0.00979 | 0 |
| 6 | 0.0015 | 0.00156 | 0.00872 | 0 |
| 7 | 0.00149 | 0.00198 | 0.00696 | 0 |
| 8 | 0.00264 | 0.00217 | 0.00928 | 0 |
| 9 | 0.00274 | 2.34E-03 | 7.95E-03 | 0 |
| 10 | 0.0041 | 0.00223 | 0.01094 | 0 |
| 11 | 0.00261 | 0.00241 | 0.01167 | 0 |
| 12 | 0.00447 | 0.00218 | 0.01675 | 0 |
| 13 | 0.00224 | 0.00263 | 0.01684 | 0 |
| 14 | 4.23E-03 | 0.00261 | 0.02051 | 0 |
| 15 | 0.00402 | 0.00265 | 0.02103 | 0 |
| 16 | 0.00402 | 0.0027 | 0.03033 | 0 |
| 17 | 0.00353 | 0.00301 | 0.02925 | 0 |
| 18 | 0.00325 | 0.00358 | 0.03665 | 0 |
| 19 | 3.71E-03 | 4.31E-03 | 0.03676 | 0 |
| 20 | 0.00489 | 0.00496 | 0.05877 | 0 |
| 21 | 0.00624 | 0.00552 | 0.06725 | 0 |
| 22 | 0.00765 | 0.00585 | 0.07505 | 0 |
| 23 | 0.0087 | 0.00604 | 0.07858 | 0 |
| 24 | 0.01025 | 0.00562 | 0.07192 | 0 |
| 25 | 0.00923 | 0.00544 | 0.0551 | 0 |
| 26 | 0.00822 | 0.00562 | 0.04326 | 0 |
| 27 | 0.00656 | 0.00652 | 0.02983 | 0 |
| 28 | 0.00875 | 0.00713 | 0.02291 | 0 |
| 29 | 7.50E-03 | 0.00872 | 0.02611 | 0 |
| 30 | 0.01031 | 0.01012 | 0.01904 | 0 |
| 31 | 0.01101 | 0.0122 | 0.01758 | 0 |
| 32 | 0.01579 | 0.0135 | 0.01635 | 0 |
| 33 | 0.01588 | 0.01591 | 0.01476 | 0 |
| 34 | 0.01934 | 0.01856 | 0.01056 | 0 |
| 35 | 0.01983 | 0.02242 | 0.01133 | 0 |
| 36 | 0.0286 | 0.0253 | 0.00949 | 0 |
| 37 | 0.02758 | 0.02933 | 0.01094 | 0 |
| 38 | 0.03456 | 0.03372 | 0.00971 | 0 |
| 39 | 0.03466 | 0.04016 | 0.0083 | 0 |
| 40 | 0.05542 | 0.04201 | 0.00574 | 0 |
| 41 | 0.06342 | 0.04205 | 0.00563 | 0 |
| 42 | 0.07077 | 0.03924 | 0.0033 | 0 |
| 43 | 0.0741 | 0.03282 | 0.00434 | 0 |
| 44 | 0.06781 | 0.02554 | 0.01588 | 0 |
| 45 | 0.05196 | 0.01983 | 0 | 0 |

(Continued)

Table 8 (continued)
Probability Mass Functions for Men at Age 20 and 40 Generated by the Full *LPd* Model with End of Period Transitions Using Richards Participation Rates*

| Years of Activity (1) | Initially Active Age 20 (2) | Initially Inactive Age 20 (3) | Initially Active Age 40 (4) | Initially Inactive Age 40 (5) |
|---|---|---|---|---|
| 46 | 0.04079 | 0.01521 | 0 | 0 |
| 47 | 0.02813 | 0.01337 | 0 | 0 |
| 48 | 0.02161 | 0.01263 | 0 | 0 |
| 49 | 0.02462 | 0.01032 | 0 | 0 |
| 50 | 0.01795 | 0.00946 | 0 | 0 |
| 51 | 0.01658 | 0.00857 | 0 | 0 |
| 52 | 0.01541 | 0.00745 | 0 | 0 |
| 53 | 0.01392 | 0.0064 | 0 | 0 |
| 54 | 0.00996 | 0.00619 | 0 | 0 |
| 55 | 0.01068 | 0.00566 | 0 | 0 |
| 56 | 0.00895 | 0.00553 | 0 | 0 |
| 57 | 0.01032 | 0.00501 | 0 | 0 |
| 58 | 0.00915 | 0.00397 | 0 | 0 |
| 59 | 0.00783 | 0.0035 | 0 | 0 |
| 60 | 0.00541 | 0.00309 | 0 | 0 |
| 61 | 0.00531 | 0.00261 | 0 | 0 |
| 62 | 0.00311 | 0.00369 | 0 | 0 |
| 63 | 0.00409 | 0.00372 | 0 | 0 |
| 64 | 0.01498 | 0 | 0 | 0 |
| Total | 1 | 1 | 1 | 1 |
| WLE | 40.6 | 25 | 22.3 | 0 |

*Participation Rates from Richards and Abele, *Life and Worklife Expectancies*, Lawyers and Judges Publishing Co., 1999.

Finally, since the *LPd* model is nested within the more general Markov model, its restrictions are statistically testable. We need not do the formal econometrics tests to believe that the demographic/actuarial model, *CVa* model, or the *LPi* model to follow, will fail hypothesis tests for the very reason that conventional models were abandoned over 20 years ago by the BLS: ${}^{i}p_x^a$ will exceed 0 in some post-peak years. Even the *LPd* model implies too high a ${}^{a}p_x^a$ value in all years. Nevertheless there will be situations where no better transition data are available, and the demographic or *LPd* model will be the best available model by default. One such situation would be when issues of substandard health predominate. Another instance appears to be railroad workers in FELA cases, where seniority resignation would suggest that the "once out, forever out" assumption would apply to such late age workers.

As is true of any Markov model, once ${}^{a}e_x^a$ and ${}^{i}e_x^a$ are computed, the economy-wide worklife expectancy may be computed as

$$(31) \qquad {}^{\bullet}e_x^a = pp_x({}^{a}e_x^a) + (1 - pp_x)({}^{i}e_x^a)\,.$$

Table 7 illustrates *LPd* calculations for a dataset taken from Richards and Abele (1999). Columns (2) and (3) compute the worklife expectancies starting active and starting inactive, ${}^{a}e_x^a$ and ${}^{i}e_x^a$, for all ages with the *LPd* model. Notice that at the peak age of 27 and beyond, ${}^{i}e_x^a = 0$, as is usual for the *CVi* model. The fourth column computes ${}^{\bullet}e_x^a$, and it is shown to equal the demographic/actuarial recipe quantity at all ages. Column (6) shows that these demographic method calculations replicate those published in Richards and Abele, as we discussed. The final column shows the substantial agreement between ${}^{a}e_x^a$ in column (2) and worklife expectancy for actives using the BLS method.

Table 8 provides the mass functions at ages 20 and 40 for those initially active and initially inactive in the *LPd* model. There are no negative probabilities in these functions (*i.e.* they are *bona fide* mass functions), since the Richards data set satisfies an Extended MC: $pp_{x+1}$ rises before the peak and falls after the peak. The mathematical expectations of those mass functions are computed at the bottom of the table and agree with the corresponding entries in Table 7.

We can directly demonstrate the *LPd* model's ability to reproduce demographic model worklife expectancies for actives and the entire population, at or beyond peak participation, by noting from (16) that

$$(32)\ p_{YFS}(x,a,t;CVa) = \frac{L_{x+t-1}}{L_x}\frac{pp_{x+t-1}}{pp_x} - \frac{L_{x+t}}{L_x}\frac{pp_{x+t}}{pp_x} = \frac{L_{x+t-1}pp_{x+t-1} - L_{x+t}pp_{x+t}}{L_x pp_x},\ t > 0$$

and

$$p_{YFS}(x,a,0;CVa) = 0\ .$$

The expected value of years until final separation (or years of activity) is

$$
\begin{aligned}
(33)\quad E[T] = {}^{a}e_x^a &= p_{YFS}(x,a,0;CVa)(0) + \sum_{t=1}^{TA-x}\frac{L_{x+t-1}pp_{x+t-1} - L_{x+t}pp_{x+t}}{L_x pp_x}t \\
&= \frac{L_x pp_x - L_{x+1}pp_{x+1}}{L_x pp_x}(1) + \frac{L_{x+1}pp_{x+1} - L_{x+2}pp_{x+2}}{L_x pp_x}(2) + \\
&\quad \frac{L_{x+2}pp_{x+2} - L_{x+3}pp_{x+3}}{L_x pp_x}(3) + \ldots + \frac{L_{TA-1}pp_{TA-1} - L_{TA}pp_{TA}}{L_x pp_x}(TA-x) \\
&= \frac{L_x pp_x}{L_x pp_x} + \frac{L_{x+1}pp_{x+1}}{L_x pp_x} + \frac{L_{x+2}pp_{x+2}}{L_x pp_x} + \ldots + \frac{L_{TA-1}pp_{TA-1}}{L_x pp_x} - \frac{L_{TA}pp_{TA}}{L_x pp_x}(TA-x) \\
&= \sum_{t=0}^{TA-x-1}\frac{L_{x+t}}{L_x}\frac{pp_{x+t}}{pp_x}\quad [L_{TA} = 0 \text{ since there are no survivors at age } TA]
\end{aligned}
$$

$$(33a)\quad = \frac{\sum_{t=0}^{TA-x-1} L_{x+t}pp_{x+t}}{L_x pp_x} = \frac{Tw_x}{L_x pp_x} \cong ew'_x = WLE(CVa)$$

using the notation in (5) and noting that age $TA = 111$.

In (33a) we observe that $(Tw_x / L_x pp_x)$ is approximately equal to the demographic worklife expectancy (5) for initial actives, with the modification that $L_x pp_x$ appears in the denominator rather than $l_x pp_x$. Since $L_x$ and $l_x$ are typically quite close in value,[16] we expect very close agreement (at peak and post peak participation ages) between worklife expectancy calculated with the *LPd* model and the demographic approach as is shown in Table 7.

From (31) and (33a), worklife for the entire population is

$$(34)\qquad {}^{\bullet}e_x^a = pp_x({}^a e_x^a) + (1 - pp_x)({}^i e_x^a) = pp_x \frac{Tw_x}{L_x pp_x} + (1 - pp_x)(0) = \frac{Tw_x}{L_x} \cong ew_x = WLE(CVm_x)$$

in (4) for peak and post-peak ages.

## VI. The *LPi* Model

We discuss the *LPi* model generally before presenting transition probabilities for it within the context of mid-period transitions.[17] In this model, a person's current labor force status while alive, has no affect on status in the next period. The model therefore imposes a restriction on the transition probabilities of the general Markov model, which is maintained. Most succinctly, the *LPi* model specifies:

$$(35)\qquad {}^a p_x^a = {}^i p_x^a \text{ (and hence } {}^a p_x^i = {}^i p_x^i \text{) for all ages } x.$$

Returning to the notation of Section II where age is again denoted by $t$ and the state, *a,i*, or *d* at age $t$ is denoted by $X_t$, consider the claim that:

$$(36)\qquad P[X_{t+2}, X_{t+1} \mid X_t = a] = P[X_{t+2}, X_{t+1} \mid X_t = i].$$

To prove this, nine equalities must hold. One of these is

$P[X_{t+2} = a, X_{t+1} = a \mid X_t = a] = P[X_{t+2} = a, X_{t+1} = a \mid X_t = i]$. The term on the left hand side is ${}^a p_{x+1}^a \, {}^a p_x^a$ and the term on the right hand side is ${}^a p_{x+1}^a \, {}^i p_x^a$. We have used the Markov property when we made the substitution $P[X_{t+2,} = a \mid X_{t+1} = a, X_t = a] = P[X_{t+2} = a \mid X_{t+1} = a] = {}^a p_{x+1}^a$ on the left hand side, and the similar substitution on the right hand side. From (35) these are evidently equal.

There is no reason to limit the joint distribution to two future $(X_{t+2,} X_{t+1})$ events: the Markov property and the *LPi* assumption (35) imply

---

[16]For example, at age 40, $L_{40} = .999 l_{40}$. Even at age 73, the last age shown in Table 7, $L_{73} = .979 l_{73}$; and, when worklife is rounded to a tenth of a year, the worklife is the same when calculated with the *LPd* and the demographic approaches.

[17]We illustrate the *LPi* model with mid-period transitions for two reasons: (1) The resulting worklife expectancies are closest to those produced with the demographic approach. (2) The accompanying tables illustrating the *LPi* model provide the first set of detailed characteristics (*i.e.*, several measures of central tendency, shape, and probability intervals) for this model—tables which are more directly comparable to other work by SC (2001a, 2003) that also uses mid-period transitions.

that the probability of *any future sample path* is independent of the starting *labor force* state:

(37) $P[X_{t+s}, X_{t+s-1}, ..., X_{t+2,} X_{t+1} \mid X_t = a] = P[X_{t+s}, X_{t+s-1}, ..., X_{t+2,} X_{t+1} \mid X_t = i]$, all *s*.

The equations (35) are a succinct statement of the *LPi* model, but the stronger statement (37) is useful because it says that any future sample has a probability which does not depend on whether one is active or inactive.

Let ${}^a p_x^a = {}^i p_x^a = p_x^a$ and ${}^i p_x^i = {}^a p_x^i = p_x^i$ express the *LPi* model's common restrictions. Further, the proof showed that the probability of any sample path is computed as the product of transition probabilities (the Markov model) which in turn are each independent of the beginning state. For example,

(38) $P[X_{t+3} = a, X_{t+2} = i, X_{t+1} = a \mid X_t = a] = p_{x+2}^a p_{x+1}^i p_x^a = P[X_{t+3} = a, X_{t+2} = i, X_{t+1} = a \mid X_t = i]$

While this equation presents a useful computing formula, the factorization of the joint probabilities suggests statistical independence. The $X_t$ are not conditionally independent as random variables, however. For example,

$$P[X_{t+2} = a, X_{t+1} = d \mid X_t = a] = 0 \neq P[X_{t+2} = a \mid X_t = a] P[X_{t+1} = d \mid X_t = a]$$

since each term in the right hand side factorization is non-zero. Nor is there unconditional independence, since

$$P[X_{t+1} = a, X_t = a] = P[X_{t+1} = a \mid X_t = a] P[X_t = a] = {}^a p_x^a P[X_t = a]$$

does not generally equal $P[X_{t+1} = a] P[X_t = a]$ because ${}^a p_x^a \neq P[X_{t+1} = a]$ - the former is a transition probability, and the latter unconditional probability depends on the initial conditions of the process. Another counterexample is $P[X_{t+1} = a, X_t = d] = 0 \neq P[X_{t+1} = a] P[X_t = d]$, where the latter two probabilities are non-zero (and dependent on initial conditions).

The *i* in *LPi* thus refers to the independence of sample path probabilities with respect to the initial labor force state. Despite the simplified computing formula (38), there are no simple closed form expressions for the YA or YFS probability mass functions in this model. In the notation we have been using, the transition probabilities in this model are:

(39) $${}^a p_x^a = {}^i p_x^a = (L_{x+1}/L_x) pp_x, \; {}^a p_x^i = {}^i p_x^i = (L_{x+1}/L_x)(1 - pp_x),$$
$${}^a p_x^d = {}^i p_x^d = 1 - L_{x+1}/L_x$$

The *LP* part of our name will now be related to the *LPE* model whose dominant feature involves expressions such as $P[X_{t+j} = a \mid living \ at \ t]$. To be active at age *t+j* one must survive an additional *j* years; and the probability of activity at *t+j* is independent of status at age *t+j*–1. Applying the conditioning rules of the

*LPi* model, and employing telescoping cancellations similar to those in the *LPd* model, the standard *LPE* result emerges:

(40)
$$P[X_{t+j} = a \mid living\ \ at\ \ t]$$
$$= P[X_{t+j} = a \mid living\ \ at\ \ t+j-1]P[living\ \ at\ \ t+j-1 \mid living\ \ at\ \ t+j-2]$$
$$\ldots P[living\ \ at\ \ t+1 \mid living\ \ at\ \ t] = \frac{L_{t+j}}{L_{t+j-1}} pp_{t+j} \frac{L_{t+j-1}}{L_{t+j-2}} \ldots \frac{L_{t+1}}{L_t} = \frac{L_{t+j}}{L_t} pp_{t+j}$$

To the terms "*L*" and "*P*" on the right hand side of this last equation, *LPE* practitioners multiply by the probability of employment. The equation as written gives the contribution between ages *t+j-1* and *t+j* to "worklife expectancy" in the *LPE* model. Summing over *j* gives the notion of "worklife expectancy" embedded in that model. Consequently the pmf recursion for years of activity given below, and associated tabulations, will provide the same statistics for YA and YFS for the *LPE* model as we have provided earlier for the general Markov model, once the "E" is dropped, so that standard worklife expectancy as time in the labor force is being measured.

To be able to provide so much more analysis for the *LPE* model, we needed to develop this machinery—to stochasticize or complete it, in the same way that the previous section stochasticized the conventional model. The *LPE* model as formulated and practiced heretofore has been specified in terms of means only.

In fact, the contribution to worklife expectancy between arbitrary ages *t+j*-1 and *t+j* is, except for an inconsequential $L_x$ versus $l_x$ distinction in the base age *x*, the same contribution to worklife expectancy as in the conventional or actuarial/demographic model, despite the language of the conventional model in (4) and (5) appearing different. The *LPi* model, in not distinguishing between those starting active and inactive, treat these groups identically, whereas the *CVa* model allocated all of the $(L_{t+j}/L_t)pp_{t+j}$ years to those starting active and nothing to those starting inactive. The difference in these models, observationally equivalent from the point of view of the economy wide average, is in the way they divide the same reported level of activity.

In the *LPi* model for years of activity with transitions at mid year, the pmf for initial actives and initial inactives are related by

$$p_{YA}\left(x,a,t;LPi\right) = p_{YA}(x,i,t-.5;LPi),\ t \geq .5,\ \ p_{YA}\left(x,a,0;LPi\right) = 0\ .$$

That is, the graph of the pmf for initial actives is simply the pmf graph for initial inactives but translated .5 units (one-half year) to the right. One may prove this by induction from the recursions for the pmf given below. Alternatively, if one starts active, and the transition is at mid-year, one is credited with a half-year of activity from the transition convention. After the first half-year, we have seen that the sample paths have the same probabilities independently of the starting state, so the probability of *s* additional years of activity is the same from this point forward. For any sample path yielding *s* additional years from the mid-point, the person starting inactive will experience $s$ years total, while the person starting active will experience $s$+.5 additional years of activity.

The pmf for the $LPi$ model for years of activity with mid-period transitions is given by the recursions (41a)-(41f), where the upper left superscript on any transition probability may be suppressed, as indicated above:

Probability Mass Function for the $LPi$ Model for Years of Activity with Mid-Period Transitions

(41a) $p_{YA}\left(TA,a,0;LPi\right)=1$ and $p_{YA}\left(TA,i,0;LPi\right)=1$

(41b) $p_{YA}\left(x,a,0;LPi\right)=0,\ x\le TA\text{-}1,$

(41c) $p_{YA}\left(x,a,.5;LPi\right)={}^{a}p_x^{i}\,p_{YA}(x+1,i,0;LPi)+{}^{a}p_x^{d}\ ,\ x\le TA\text{-}1,$

(41d) $p_{YA}\left(x,i,0;LPi\right)={}^{i}p_x^{i}\,p_{YA}(x+1,i,0;LPi)+{}^{i}p_x^{d}\ ,\ x\le TA\text{-}1,$

(41e) $p_{YA}\left(x,a,t;LPi\right)={}^{a}p_x^{a}\,p_{YA}(x+1,a,t-1;LPi)+{}^{a}p_x^{i}\,p_{YA}(x+1,i,t-.5;LPi)\ ,\ x<TA$ and $t>1,$

(41f) $p_{YA}\left(x,i,t;LPi\right)={}^{i}p_x^{a}\,p_{YA}(x+1,a,t-.5;LPi)+{}^{i}p_x^{i}\,p_{YA}(x+1,i,t;LPi)\ ,\ x<TA$ and $t\ge 1$

Here (41a) is a global condition, (41b)—(41d) are boundary conditions, and (41e)—(41f) are the main recursions.

Probability Mass Function for the $LPi$ Model for Years until Final Separation with Mid-Period Transitions

(42a) $p_{YFS}\left(TA,a,0;LPi\right)=1$ and $p_{YFS}\left(TA,i,0;LPi\right)=1$

(42b) $p_{YFS}\left(x,a,0;LPi\right)=0,\ x\le TA\text{-}1,$

(42c) $p_{YFS}\left(x,a,.5;LPi\right)={}^{a}p_x^{i}\,p_{YFS}(x+1,i,0;LPi)+{}^{a}p_x^{d}\ ,\ x\le TA\text{-}1,$

(42d) $p_{YFS}\left(x,i,0;LPi\right)={}^{i}p_x^{i}\,p_{YFS}(x+1,i,0;LPi)+{}^{i}p_x^{d}\ ,\ x\le TA\text{-}1,$

(42e) $p_{YFS}\left(x,i,1;LPi\right)=0,\ x\le TA\text{-}1,$

(42f) $p_{YFS}\left(x,a,t;LPi\right)={}^{a}p_x^{a}\,p_{YFS}(x+1,a,t-1;LPi)+{}^{a}p_x^{i}\,p_{YFS}(x+1,i,t-1;LPi)\ ,\ x<TA$ and $t>1,$

(42g) $p_{YFS}\left(x,i,t;LPi\right)={}^{i}p_x^{a}\,p_{YFS}(x+1,a,t-1;LPi)+{}^{i}p_x^{i}\,p_{YFS}(x+1,i,t-1;LPi)\ ,\ x<TA$ and $t>1$

(42h) $p_{YFS}(x,a,t;LPi)=0$ , for $t=0,1,2,3,\ldots,TA-1$

(42i) $p_{YFS}(x,i,t;LPi)=0$ , for $t=.5,1,2,3,\ldots,TA-1$

The *LPi* pmf for initial actives and initial inactives for years until final separation possess the property that $p_{YFS}\left(x,a,t;LPi\right)=p_{YFS}(x,i,t;LPi)$ $t\ \geq 1$, $p_{YFS}\left(x,a,0;LPi\right)=p_{YFS}(x,i,.5;LPi)=0$, and $p_{YFS}\left(x,a,.5;LPi\right)=p_{YFS}(x,i,0;LPi)$. That is, the pmf's are the same for initial actives and inactives for one or more years until final separation; there is a zero probability of zero (.5) years until final separation for initial actives (inactives); and the probability of one-half year until final separation for initial actives is the same as zero years for initial inactives. The pmf for the *LPi* model for years until final labor force separation occurs is given by (42a)-(42i): Here again (42a) is a global condition, (42b)-(42e) are boundary conditions, (42f)-(42g) are the main recursions, and (42h)-(42i) indicate values in the domain of the mass function that occur with zero probability.

Tables 9 and 10 and Figure 4 reveal the same general (but accentuated) relationship between YA and YFS as previously shown by SC but within the context of the increment-decrement model. The YFS pmf is to the right of pmf for YA, and the standard deviation and the length of probability intervals for YFS exceed that of YA. The nature of the *LPi* model magnifies these relationships. For example, the mean and standard deviation for YA for 30-year-old active men with high school in the increment-decrement model are 28.3 and 8.4 years (SC, 2001a); they are 33.7 and 10.4, respectively, for YFS (SC, 2003). The difference in means is 5.4 years (19.1% relative to YA), and the difference in standard deviations is 2.0 years (23.8%). With the *LPi* model based on Richards' participation rate data, the mean and standard deviation of YA in Table 9 are 29.5 and 6.1 years; and 38.8 and 10.3, respectively, for YFS in Table 10. Differences are 9.3 years (31.5% relative to YA) for the means and 4.2 (68.9%) years for the standard deviations.

Table 9
YA Characteristics from *LPi* Model for Initially Active Men with
High School with Mid-Period Transitions*

| | (WLE) | | | | | | Minimal 50%PI | | Inter-Quartile PI | | 10%-90% PI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age (1) | Mean (2) | Median (3) | Mode (4) | SD (5) | SK (6) | KU (7) | Low (8) | High (9) | 25% (10) | 75% (11) | 10% (12) | 90% (13) |
| 18 | 39.45 | 41.36 | 42.50 | 7.66 | -2.54 | 10.57 | 40.00 | 44.91 | 38.41 | 43.64 | 31.62 | 45.51 |
| 19 | 38.81 | 40.69 | 41.50 | 7.53 | -2.53 | 10.50 | 39.00 | 43.82 | 37.79 | 42.93 | 31.10 | 44.77 |
| 20 | 38.09 | 39.94 | 40.50 | 7.39 | -2.51 | 10.41 | 38.24 | 43.00 | 37.08 | 42.15 | 30.47 | 43.94 |
| 21 | 37.33 | 39.13 | 39.50 | 7.25 | -2.49 | 10.29 | 37.27 | 42.00 | 36.29 | 41.33 | 29.81 | 43.10 |
| 22 | 36.52 | 38.28 | 39.50 | 7.10 | -2.46 | 10.15 | 36.29 | 41.00 | 35.46 | 40.46 | 29.11 | 42.24 |
| 23 | 35.68 | 37.39 | 38.50 | 6.96 | -2.43 | 9.99 | 36.00 | 40.66 | 34.60 | 39.56 | 28.37 | 41.33 |
| 24 | 34.83 | 36.49 | 37.50 | 6.81 | -2.39 | 9.81 | 35.00 | 39.62 | 33.74 | 38.64 | 27.62 | 40.42 |
| 25 | 33.96 | 35.58 | 36.50 | 6.68 | -2.36 | 9.64 | 34.00 | 38.59 | 32.86 | 37.71 | 26.86 | 39.48 |
| 26 | 33.08 | 34.65 | 35.50 | 6.55 | -2.33 | 9.46 | 33.00 | 37.56 | 31.98 | 36.78 | 26.08 | 38.54 |
| 27 | 32.20 | 33.73 | 34.50 | 6.43 | -2.29 | 9.29 | 32.00 | 36.54 | 31.07 | 35.84 | 25.27 | 37.60 |
| 28 | 31.31 | 32.80 | 33.50 | 6.31 | -2.26 | 9.12 | 31.00 | 35.52 | 30.15 | 34.90 | 24.47 | 36.65 |
| 29 | 30.42 | 31.88 | 32.50 | 6.19 | -2.23 | 8.95 | 30.51 | 35.00 | 29.23 | 33.95 | 23.68 | 35.70 |
| 30 | 29.54 | 30.96 | 31.50 | 6.07 | -2.19 | 8.78 | 29.54 | 34.00 | 28.32 | 33.01 | 22.91 | 34.75 |

(Continued)

Table 9 (continued)
YA Characteristics from *LPi* Model for Initially Active Men with
High School with Mid-Period Transitions*

| | (WLE) | | | | | | Minimal 50%PI | | Inter-Quartile PI | | 10%-90% PI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age (1) | Mean (2) | Median (3) | Mode (4) | SD (5) | SK (6) | KU (7) | Low (8) | High (9) | 25% (10) | 75% (11) | 10% (12) | 90% (13) |
| 31 | 28.66 | 30.03 | 30.50 | 5.94 | -2.15 | 8.61 | 28.57 | 33.00 | 27.42 | 32.09 | 22.12 | 33.80 |
| 32 | 27.78 | 29.11 | 29.50 | 5.81 | -2.12 | 8.43 | 27.60 | 32.00 | 26.52 | 31.16 | 21.34 | 32.85 |
| 33 | 26.91 | 28.19 | 28.50 | 5.68 | -2.08 | 8.25 | 26.64 | 31.00 | 25.63 | 30.23 | 20.57 | 31.90 |
| 34 | 26.03 | 27.27 | 27.50 | 5.55 | -2.03 | 8.07 | 25.67 | 30.00 | 24.74 | 29.30 | 19.82 | 30.95 |
| 35 | 25.16 | 26.35 | 26.50 | 5.42 | -1.99 | 7.88 | 24.70 | 29.00 | 23.86 | 28.37 | 19.07 | 30.00 |
| 36 | 24.29 | 25.43 | 26.50 | 5.29 | -1.95 | 7.69 | 23.73 | 28.00 | 23.00 | 27.45 | 18.30 | 29.08 |
| 37 | 23.43 | 24.52 | 25.50 | 5.16 | -1.90 | 7.50 | 23.00 | 27.23 | 22.10 | 26.52 | 17.55 | 28.16 |
| 38 | 22.56 | 23.61 | 24.50 | 5.02 | -1.85 | 7.30 | 22.00 | 26.19 | 21.20 | 25.59 | 16.82 | 27.24 |
| 39 | 21.70 | 22.70 | 23.50 | 4.89 | -1.80 | 7.10 | 21.00 | 25.16 | 20.31 | 24.67 | 16.09 | 26.31 |
| 40 | 20.84 | 21.79 | 22.50 | 4.75 | -1.74 | 6.90 | 20.00 | 24.13 | 19.43 | 23.74 | 15.33 | 25.38 |
| 41 | 19.99 | 20.89 | 21.50 | 4.62 | -1.69 | 6.69 | 19.00 | 23.11 | 18.55 | 22.81 | 14.60 | 24.46 |
| 42 | 19.13 | 19.99 | 20.50 | 4.48 | -1.63 | 6.47 | 18.00 | 22.09 | 17.69 | 21.89 | 13.90 | 23.53 |
| 43 | 18.29 | 19.10 | 19.50 | 4.34 | -1.56 | 6.26 | 17.96 | 22.00 | 16.84 | 20.97 | 13.17 | 22.60 |
| 44 | 17.45 | 18.20 | 18.50 | 4.21 | -1.50 | 6.03 | 17.00 | 20.97 | 16.01 | 20.07 | 12.45 | 21.68 |
| 45 | 16.61 | 17.32 | 17.50 | 4.07 | -1.43 | 5.81 | 16.00 | 19.91 | 15.14 | 19.17 | 11.77 | 20.76 |
| 46 | 15.79 | 16.44 | 16.50 | 3.93 | -1.36 | 5.58 | 15.00 | 18.86 | 14.28 | 18.28 | 11.08 | 19.83 |
| 47 | 14.96 | 15.56 | 15.50 | 3.79 | -1.28 | 5.36 | 14.00 | 17.81 | 13.44 | 17.39 | 10.37 | 18.91 |
| 48 | 14.14 | 14.69 | 15.50 | 3.65 | -1.20 | 5.12 | 13.24 | 17.00 | 12.61 | 16.50 | 9.69 | 17.99 |
| 49 | 13.33 | 13.83 | 14.50 | 3.52 | -1.12 | 4.89 | 12.29 | 16.00 | 11.81 | 15.62 | 9.05 | 17.11 |
| 50 | 12.52 | 12.98 | 13.50 | 3.38 | -1.02 | 4.65 | 11.34 | 15.00 | 11.02 | 14.73 | 8.33 | 16.24 |
| 51 | 11.72 | 12.13 | 12.50 | 3.24 | -0.93 | 4.41 | 10.38 | 14.00 | 10.18 | 13.84 | 7.68 | 15.36 |
| 52 | 10.93 | 11.28 | 11.50 | 3.11 | -0.82 | 4.17 | 10.00 | 13.55 | 9.37 | 12.96 | 7.05 | 14.48 |
| 53 | 10.14 | 10.44 | 10.50 | 2.97 | -0.72 | 3.93 | 9.00 | 12.47 | 8.58 | 12.11 | 6.36 | 13.60 |
| 54 | 9.37 | 9.61 | 9.50 | 2.84 | -0.61 | 3.71 | 8.00 | 11.40 | 7.83 | 11.27 | 5.74 | 12.72 |
| 55 | 8.61 | 8.81 | 9.50 | 2.70 | -0.49 | 3.50 | 7.65 | 11.00 | 7.09 | 10.44 | 5.13 | 11.84 |
| 56 | 7.86 | 8.02 | 8.50 | 2.58 | -0.37 | 3.31 | 6.76 | 10.00 | 6.32 | 9.61 | 4.49 | 10.96 |
| 57 | 7.13 | 7.25 | 7.50 | 2.45 | -0.26 | 3.14 | 5.87 | 9.00 | 5.61 | 8.79 | 3.97 | 10.16 |
| 58 | 6.43 | 6.50 | 6.50 | 2.33 | -0.14 | 3.01 | 5.00 | 8.02 | 4.96 | 7.99 | 3.35 | 9.38 |
| 59 | 5.75 | 5.78 | 5.50 | 2.21 | -0.03 | 2.92 | 4.05 | 7.00 | 4.28 | 7.26 | 2.86 | 8.60 |
| 60 | 5.10 | 5.10 | 5.50 | 2.09 | 0.08 | 2.86 | 4.00 | 6.85 | 3.66 | 6.54 | 2.33 | 7.81 |
| 61 | 4.50 | 4.47 | 4.50 | 1.97 | 0.18 | 2.83 | 3.00 | 5.70 | 3.12 | 5.83 | 1.94 | 7.07 |
| 62 | 3.96 | 3.89 | 3.50 | 1.86 | 0.28 | 2.85 | 2.42 | 5.00 | 2.60 | 5.22 | 1.50 | 6.47 |
| 63 | 3.49 | 3.40 | 3.50 | 1.75 | 0.37 | 2.88 | 2.00 | 4.44 | 2.20 | 4.67 | 1.21 | 5.84 |
| 64 | 3.09 | 2.97 | 2.50 | 1.64 | 0.45 | 2.94 | 1.71 | 4.00 | 1.86 | 4.18 | 1.02 | 5.34 |
| 65 | 2.76 | 2.63 | 2.50 | 1.54 | 0.53 | 3.02 | 1.00 | 3.17 | 1.57 | 3.78 | 0.79 | 4.86 |
| 66 | 2.48 | 2.35 | 2.50 | 1.44 | 0.60 | 3.10 | 1.03 | 3.00 | 1.36 | 3.45 | 0.63 | 4.51 |
| 67 | 2.25 | 2.09 | 1.50 | 1.35 | 0.67 | 3.20 | 1.00 | 2.85 | 1.19 | 3.11 | 0.51 | 4.09 |
| 68 | 2.03 | 1.87 | 1.50 | 1.27 | 0.74 | 3.30 | 1.00 | 2.78 | 1.05 | 2.84 | 0.43 | 3.81 |
| 69 | 1.84 | 1.68 | 1.50 | 1.18 | 0.81 | 3.42 | 0.38 | 2.00 | 0.90 | 2.62 | 0.36 | 3.56 |
| 70 | 1.68 | 1.52 | 1.50 | 1.11 | 0.88 | 3.55 | 0.51 | 2.00 | 0.77 | 2.40 | 0.31 | 3.25 |
| 71 | 1.53 | 1.37 | 0.50 | 1.03 | 0.95 | 3.70 | 0.00 | 1.37 | 0.68 | 2.16 | 0.27 | 2.95 |
| 72 | 1.39 | 1.23 | 0.50 | 0.96 | 1.03 | 3.87 | 0.00 | 1.23 | 0.60 | 1.95 | 0.24 | 2.79 |
| 73 | 1.27 | 1.09 | 0.50 | 0.89 | 1.11 | 4.06 | 0.00 | 1.09 | 0.53 | 1.82 | 0.21 | 2.62 |
| 74 | 1.17 | 0.97 | 0.50 | 0.82 | 1.20 | 4.27 | 0.02 | 0.98 | 0.48 | 1.70 | 0.19 | 2.41 |
| 75 | 1.07 | 0.88 | 0.50 | 0.75 | 1.30 | 4.53 | 0.06 | 0.94 | 0.44 | 1.57 | 0.18 | 2.15 |

*All table entries computed with the *LPi* model using Richards' participation rates.

Table 10
YFS Characteristics from *LPi* Model for Initially Active Men with
High School with Mid-Period Transitions*

| | (YFSE) | | | | | | Minimal 50% PI | | Inter-Quartile PI | | 10%-90% PI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age (1) | Mean (2) | Median (3) | Mode (4) | SD (5) | SK (6) | KU (7) | Low (8) | High (9) | 25% (10) | 75% (11) | 10% (12) | 90% (13) |
| 18 | 49.99 | 52.09 | 53.50 | 11.77 | -1.53 | 6.04 | 47.33 | 59.00 | 45.88 | 57.72 | 35.35 | 62.05 |
| 19 | 49.05 | 51.11 | 52.50 | 11.64 | -1.50 | 5.92 | 46.35 | 58.00 | 44.91 | 56.73 | 34.51 | 61.05 |
| 20 | 48.11 | 50.12 | 51.50 | 11.51 | -1.47 | 5.80 | 45.37 | 57.00 | 43.94 | 55.73 | 33.68 | 60.06 |
| 21 | 47.18 | 49.14 | 50.50 | 11.38 | -1.44 | 5.67 | 44.38 | 56.00 | 42.98 | 54.74 | 32.87 | 59.07 |
| 22 | 46.25 | 48.15 | 49.50 | 11.24 | -1.41 | 5.55 | 43.40 | 55.00 | 42.01 | 53.75 | 32.05 | 58.07 |
| 23 | 45.32 | 47.17 | 48.50 | 11.10 | -1.37 | 5.42 | 42.42 | 54.00 | 41.05 | 52.76 | 31.23 | 57.08 |
| 24 | 44.39 | 46.19 | 47.50 | 10.97 | -1.34 | 5.29 | 41.45 | 53.00 | 40.08 | 51.77 | 30.42 | 56.08 |
| 25 | 43.46 | 45.21 | 46.50 | 10.84 | -1.30 | 5.17 | 40.47 | 52.00 | 39.11 | 50.78 | 29.60 | 55.09 |
| 26 | 42.53 | 44.22 | 45.50 | 10.71 | -1.27 | 5.05 | 39.49 | 51.00 | 38.15 | 49.79 | 28.77 | 54.10 |
| 27 | 41.59 | 43.24 | 44.50 | 10.60 | -1.24 | 4.94 | 38.51 | 50.00 | 37.18 | 48.80 | 27.95 | 53.10 |
| 28 | 40.65 | 42.25 | 43.50 | 10.48 | -1.21 | 4.84 | 37.53 | 49.00 | 36.21 | 47.81 | 27.11 | 52.11 |
| 29 | 39.72 | 41.27 | 42.50 | 10.37 | -1.17 | 4.74 | 36.55 | 48.00 | 35.24 | 46.82 | 26.28 | 51.12 |
| 30 | 38.78 | 40.29 | 41.50 | 10.25 | -1.14 | 4.63 | 35.57 | 47.00 | 34.28 | 45.83 | 25.46 | 50.12 |
| 31 | 37.85 | 39.31 | 40.50 | 10.13 | -1.11 | 4.53 | 34.59 | 46.00 | 33.32 | 44.84 | 24.65 | 49.13 |
| 32 | 36.92 | 38.33 | 39.50 | 10.01 | -1.07 | 4.42 | 33.62 | 45.00 | 32.36 | 43.85 | 23.86 | 48.14 |
| 33 | 36.00 | 37.35 | 38.50 | 9.89 | -1.04 | 4.31 | 32.64 | 44.00 | 31.40 | 42.86 | 23.06 | 47.15 |
| 34 | 35.07 | 36.37 | 37.50 | 9.76 | -1.00 | 4.21 | 31.67 | 43.00 | 30.44 | 41.87 | 22.27 | 46.15 |
| 35 | 34.15 | 35.40 | 36.50 | 9.63 | -0.97 | 4.10 | 30.70 | 42.00 | 29.49 | 40.88 | 21.48 | 45.16 |
| 36 | 33.22 | 34.42 | 35.50 | 9.51 | -0.93 | 4.00 | 29.73 | 41.00 | 28.54 | 39.90 | 20.70 | 44.17 |
| 37 | 32.30 | 33.45 | 34.50 | 9.38 | -0.89 | 3.90 | 28.76 | 40.00 | 27.59 | 38.91 | 19.94 | 43.18 |
| 38 | 31.39 | 32.47 | 33.50 | 9.25 | -0.85 | 3.80 | 27.79 | 39.00 | 26.64 | 37.93 | 19.16 | 42.19 |
| 39 | 30.47 | 31.50 | 32.50 | 9.12 | -0.81 | 3.70 | 26.83 | 38.00 | 25.70 | 36.94 | 18.40 | 41.20 |
| 40 | 29.56 | 30.53 | 31.50 | 8.99 | -0.77 | 3.60 | 25.86 | 37.00 | 24.76 | 35.96 | 17.65 | 40.21 |
| 41 | 28.65 | 29.56 | 30.50 | 8.86 | -0.73 | 3.50 | 24.90 | 36.00 | 23.82 | 34.98 | 16.92 | 39.23 |
| 42 | 27.74 | 28.60 | 29.50 | 8.73 | -0.69 | 3.40 | 23.94 | 35.00 | 22.89 | 34.00 | 16.17 | 38.24 |
| 43 | 26.83 | 27.63 | 28.50 | 8.59 | -0.65 | 3.31 | 22.99 | 34.00 | 21.96 | 33.02 | 15.44 | 37.25 |
| 44 | 25.93 | 26.67 | 27.50 | 8.46 | -0.60 | 3.21 | 22.00 | 32.97 | 21.03 | 32.04 | 14.71 | 36.27 |
| 45 | 25.03 | 25.71 | 26.50 | 8.32 | -0.56 | 3.12 | 21.00 | 31.92 | 20.11 | 31.06 | 14.01 | 35.28 |
| 46 | 24.13 | 24.76 | 25.50 | 8.19 | -0.51 | 3.03 | 20.00 | 30.87 | 19.18 | 30.09 | 13.28 | 34.30 |
| 47 | 23.24 | 23.80 | 24.50 | 8.05 | -0.47 | 2.95 | 19.00 | 29.82 | 18.26 | 29.11 | 12.56 | 33.32 |
| 48 | 22.34 | 22.85 | 23.50 | 7.91 | -0.42 | 2.86 | 18.00 | 28.76 | 17.35 | 28.14 | 11.87 | 32.33 |
| 49 | 21.46 | 21.90 | 22.50 | 7.78 | -0.37 | 2.78 | 17.00 | 27.70 | 16.45 | 27.17 | 11.16 | 31.35 |
| 50 | 20.57 | 20.96 | 21.50 | 7.64 | -0.33 | 2.70 | 16.00 | 26.64 | 15.55 | 26.20 | 10.45 | 30.38 |
| 51 | 19.69 | 20.02 | 20.50 | 7.50 | -0.28 | 2.63 | 15.00 | 25.57 | 14.66 | 25.24 | 9.76 | 29.40 |
| 52 | 18.82 | 19.08 | 19.50 | 7.36 | -0.23 | 2.56 | 14.00 | 24.49 | 13.77 | 24.28 | 9.08 | 28.42 |
| 53 | 17.95 | 18.15 | 18.50 | 7.22 | -0.18 | 2.49 | 13.00 | 23.41 | 12.90 | 23.32 | 8.37 | 27.45 |
| 54 | 17.08 | 17.22 | 17.50 | 7.08 | -0.13 | 2.43 | 12.00 | 22.32 | 12.04 | 22.36 | 7.69 | 26.48 |
| 55 | 16.22 | 16.30 | 16.50 | 6.94 | -0.09 | 2.37 | 11.00 | 21.23 | 11.18 | 21.41 | 7.02 | 25.51 |
| 56 | 15.36 | 15.39 | 15.50 | 6.81 | -0.04 | 2.32 | 10.00 | 20.13 | 10.32 | 20.46 | 6.31 | 24.54 |
| 57 | 14.51 | 14.48 | 14.50 | 6.67 | 0.01 | 2.27 | 9.00 | 19.02 | 9.49 | 19.51 | 5.63 | 23.58 |
| 58 | 13.67 | 13.58 | 13.50 | 6.54 | 0.05 | 2.24 | 8.10 | 18.00 | 8.66 | 18.57 | 4.97 | 22.61 |
| 59 | 12.83 | 12.69 | 12.50 | 6.41 | 0.09 | 2.20 | 7.22 | 17.00 | 7.85 | 17.64 | 4.27 | 21.66 |
| 60 | 12.00 | 11.81 | 11.50 | 6.28 | 0.13 | 2.18 | 6.36 | 16.00 | 7.06 | 16.71 | 3.61 | 20.70 |

(Continued)

Table 10 (continued)
YFS Characteristics from *LPi* Model for Initially Active Men with
High School with Mid-Period Transitions*

| | (YFSE) | | | | | | Minimal 50% PI | | Inter-Quartile PI | | 10%-90% PI | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age (1) | Mean (2) | Median (3) | Mode (4) | SD (5) | SK (6) | KU (7) | Low (8) | High (9) | 25% (10) | 75% (11) | 10% (12) | 90% (13) |
| 61 | 11.18 | 10.94 | 10.50 | 6.14 | 0.16 | 2.16 | 5.52 | 15.00 | 6.28 | 15.78 | 2.97 | 19.75 |
| 62 | 10.37 | 10.09 | 9.50 | 6.01 | 0.20 | 2.15 | 4.68 | 14.00 | 5.52 | 14.87 | 2.31 | 18.81 |
| 63 | 9.58 | 9.24 | 0.50 | 5.86 | 0.24 | 2.14 | 0.00 | 9.24 | 4.78 | 13.96 | 1.68 | 17.87 |
| 64 | 8.80 | 8.41 | 0.50 | 5.70 | 0.29 | 2.13 | 0.00 | 8.41 | 4.06 | 13.06 | 1.07 | 16.93 |
| 65 | 8.05 | 7.58 | 0.50 | 5.51 | 0.35 | 2.13 | 0.00 | 7.58 | 3.33 | 12.17 | 0.79 | 16.00 |
| 66 | 7.33 | 6.77 | 0.50 | 5.31 | 0.41 | 2.15 | 0.00 | 6.77 | 2.62 | 11.28 | 0.63 | 15.08 |
| 67 | 6.63 | 5.96 | 0.50 | 5.09 | 0.48 | 2.18 | 0.00 | 5.96 | 1.93 | 10.40 | 0.51 | 14.16 |
| 68 | 5.97 | 5.17 | 0.50 | 4.85 | 0.56 | 2.24 | 0.00 | 5.17 | 1.24 | 9.53 | 0.43 | 13.25 |
| 69 | 5.34 | 4.39 | 0.50 | 4.60 | 0.65 | 2.32 | 0.00 | 4.39 | 0.90 | 8.66 | 0.36 | 12.34 |
| 70 | 4.74 | 3.63 | 0.50 | 4.32 | 0.75 | 2.44 | 0.00 | 3.63 | 0.77 | 7.81 | 0.31 | 11.44 |
| 71 | 4.19 | 2.88 | 0.50 | 4.03 | 0.86 | 2.60 | 0.00 | 2.88 | 0.68 | 6.96 | 0.27 | 10.55 |
| 72 | 3.67 | 2.16 | 0.50 | 3.73 | 0.97 | 2.80 | 0.00 | 2.16 | 0.60 | 6.14 | 0.24 | 9.66 |
| 73 | 3.20 | 1.45 | 0.50 | 3.42 | 1.10 | 3.05 | 0.00 | 1.45 | 0.53 | 5.33 | 0.21 | 8.78 |
| 74 | 2.76 | 0.97 | 0.50 | 3.10 | 1.24 | 3.38 | 0.02 | 0.98 | 0.48 | 4.52 | 0.19 | 7.91 |
| 75 | 2.37 | 0.88 | 0.50 | 2.77 | 1.39 | 3.78 | 0.06 | 0.94 | 0.44 | 3.73 | 0.18 | 7.04 |

*All table entries computed with the *LPi* model using Richards' participation rates.

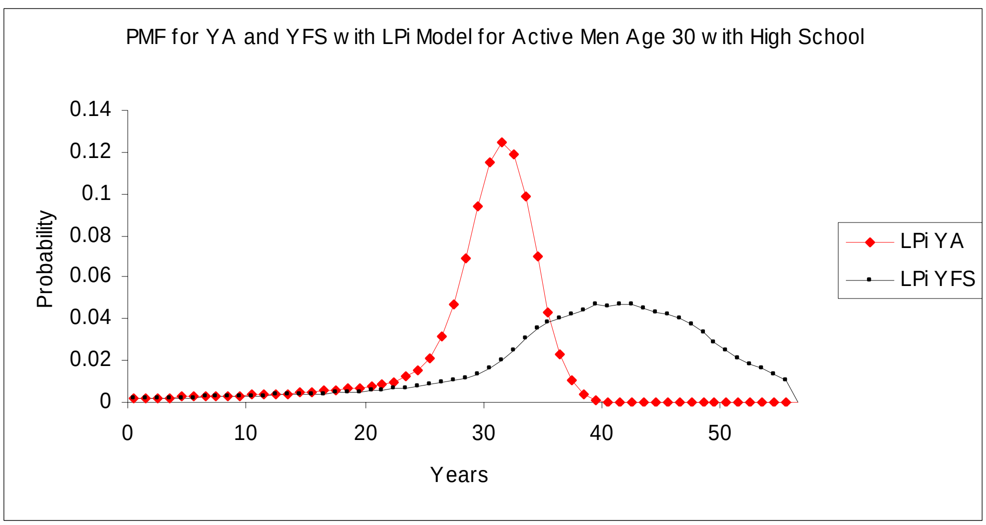


Figure 4
Probability Mass Functions for *YA* and *YFS* from *LPi* Model for
Initially Active Men Age 30 with High School

## VI. Conclusion

*The conventional model of worklife has been shown be a special case of the Markov model.* Reanalysis of the old conventional model, after over 20 years and with modern methods, reveals two restricted Markov models

(*CV*/demographic/actuarial/Richards and *CVa*), with the possibility of extension to the *CVi* and the *LPd* models. Special cases of the *CVa* model provide the only known setting where the HPR and the Nelson-type estimators make statistical sense, but within this setting, years to final separation and years of activity are the same thing. For there to even *be* a probability distribution associated with the various estimators, strong regularity conditions have been uncovered. When present, methods have been offered which permit computation of, say, the standard error of any of these models, and the tabulation of the same statistics as are available for the usual, unrestricted Markov model. *However, in the light of the further theoretical and empirical problems discussed here, and the original BLS reasons articulated in Bulletin 2135, the conventional model and its variants should properly remain abandoned in favor of the more general Markov model, provided data are available which permit the estimation of the latter.*

More promising but unfortunately no better supported by the data is the *LPi* model, which may be implicit in the use of *LPE* models. While those users should welcome the recursions provided here and the concomitant ability to move beyond simply reporting means, the fact remains that there is useful information in today's state since ${}^{a}p_x^a \neq {}^{i}p_x^a$ . *Therefore, there is every reason to embrace the Markov model until it too is dominated by a superior model.*

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