Arithmetic Means, Geometric Means,
Accumulation Functions, and Present Value Functions

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I. Introduction

There are "well known" relations between the arithmetic mean of a random variable and the geometric mean. Part of this paper is offered with the belief that perhaps some of these relations are not as well known and transparent as is sometimes assumed. In addition to asserting several relations, we provide proofs of propositions with the thought that forensic economists engaged in personal injury and business valuation litigation will find it useful to have a single and convenient reference for concepts and propositions involving the arithmetic and geometric mean. This paper also deals with a proposition and an often-cited example that has been proffered to show that the arithmetic mean of returns on an investment, when compounded for multiple time periods, gives the expected value of wealth. We examine this proposition and show sufficient conditions under which it is correct. Finally, and perhaps of most interest to forensic economists, this paper examines the expected present value function when discounting with the arithmetic mean and geometric mean. In this paper, future returns are viewed as random variables rather than constants, as is usually the case.

Section II of the paper begins with the definitions of the arithmetic and geometric means, the accumulation of wealth from returns, and the present value of one dollar. Here we specify a growth-rate random variable and its probability distribution, and we use the probability distribution to define the arithmetic and geometric means, accumulated wealth and present value. Next, we give the definitions of estimators of the arithmetic and geometric means, wealth accumulation, and present value. Then we give definitions using a particular realization of returns. This section of the paper contains several propositions stated as remarks which are proved in Appendix A. Section III of the paper considers an example of wealth accumulation. Section IV deals with the present value function and the use of the arithmetic mean and geometric mean in discounting. We make concluding comments in Section V.

II. Arithmetic and Geometric Means, Expected Wealth, and Present Value

Growth Rates from a Known Probability Distribution

Let R be a random variable of growth rates that satisfies the restriction $R \geq -1$. Assume that R takes on the values $r_1, r_{2,} \ldots, r_m$ with probabilities $p_1, p_2, \ldots p_m$. We define the arithmetic and geometric means as A and G, respectively.[1]

[1] We could, of course, define A more simply as $A = \sum_{i=1}^{m} p_i r_i$ which is equivalent to formula (1). We choose to use formula (1) because the terms $(1+r_i)$ *i=1,2,....,m* are nonnegative, and the definition of A given in formula (1) is more easily comparable to the definition of G in formula (2).

(1) $A = E(R) = \mu_R = \sum_{i=1}^{m} p_i(1+r_i) - 1$

(2) $G = \prod_{i=1}^{m} (1+r_i)^{p_i} - 1$

The expected accumulated value of one dollar of wealth after *n* periods (assuming realizations of *R* are independent) is

(3) $E(W) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} p_{i_1}p_{i_2}\cdots p_{i_n}(1+r_{1i_1})(1+r_{2i_2})\cdots(1+r_{ni_n})$.

The expected present value of one dollar to be received in *n* periods (assuming realizations of *R* are independent) is

(4) $E(PVF) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} p_{i_1}p_{i_2}\cdots p_{i_n}\frac{1}{(1+r_{1i_1})(1+r_{2i_2})\cdots(1+r_{ni_n})}$

Remarks 1, 2, 3, and *4* which follow immediately below, deal with *A*, *G*, *W*, and *PVF* as they are defined in formulas (1), (2), (3), and (4). These remarks are based on knowing the exact distribution of returns *R*, and they are proved in the Appendix A. In order, the remarks say: the arithmetic mean is the average value of the growth rate random variable, the geometric mean is equal to or smaller than the arithmetic mean, an investment growing at the arithmetic mean equals the expected value of accumulated wealth, and the expected present value of a future amount of one dollar is computed with the mean[2] of the random variable $\frac{1}{(1+R)}$.

Remark 1 The arithmetic mean return is the expected value of the random variable *R*, *i.e.*, $A = E[R] = \mu_R$.

Remark 2 The geometric mean is less than or equal to the arithmetic mean, *i.e.*, $G \le A = \mu_R$.

Remark 3 One dollar of initial wealth has an expected value after *n* periods equal to one dollar compounded at the arithmetic mean for *n* periods, *i.e.*, $E(W) = (1+\mu_R)^n = (1+A)^n$.

[2] Technically, for R discrete, we must assume that there is 0 probability that R= -1, since, if this is not the case, $E\left(\frac{1}{(1+R)}\right)$ is not defined. A similar remark would apply to the density of R around -1 in the continuous case.

Remark 4 One dollar of wealth to be received *n* periods in the future has an expected present value equal to $E(PVF) = \mu^n_{(1/(1+R))}$ where $\mu^n_{(1/(1+R))}$ is the expected value of the random variable $\frac{1}{(1+R)}$.

Growth Rates Drawn from a Random Sample

Since we do not know the exact distribution of R (except perhaps in hypothetical examples used for illustrative purposes), we consider a random sample $R_1, R_2, \ldots, R_n$ taken at one point in time or over *n* future time periods from the distribution of R. Formulae (5)-(8) define estimators of the arithmetic mean, geometric mean, wealth accumulation, and present value from such a random sample taken at one point in time. We also can think of $R_1, \ldots, R_n$ as a time series of future returns which are independent, and identically distributed, random variables defined over *n* future time periods. In that case, formulae (5)-(8) represent estimators of the means of returns corresponding to the arithmetic mean, geometric mean, wealth accumulation, and present value over time.

(5) $$A_n = \frac{1}{n}\sum_{i=1}^{n}(1+R_i) - 1$$

(6) $$G_n = \left[\prod_{i=1}^{n}(1+R_i)\right]^{\frac{1}{n}} - 1$$

The estimator of the accumulated value of one dollar of wealth after *n* periods is

(7) $$W_n = \prod_{i=1}^{n}(1+R_i).$$

The estimator of the present value of one dollar of future wealth to be received *n* periods in the future is

(8) $$PVF_n = \prod_{i=1}^{n}\left(\frac{1}{(1+R_i)}\right).$$

Remarks 5, 6, 7, and 8 (proved in the Appendix A) deal with the estimators of the arithmetic mean, geometric mean, wealth, and present value as defined in formulas (5), (6), (7), and (8). In order, these remarks say that the arithmetic mean from a random sample is an unbiased estimator of the population mean, the expected value of the geometric mean computed from a random sample equals or is less than the expected value of the sample arithmetic mean, expected wealth equals an investment growing each period at the arithmetic mean, and expected present value is computed with the mean of the $\frac{1}{(1+R)}$ random variable. *Remark 9* says that expected present value exceeds the present value when the arithmetic mean is used in discounting.

Remark 5 The expected value of the estimator of the arithmetic mean is equal to the (earlier encountered) expected value of the random variable of growth rates R, *i.e.*, $E[A_n] = \mu_R$.(This result does not require independence of the R_i.)

Remark 6 The expected value of the estimator of the geometric mean is less than or equal to the expected value of the estimator of the arithmetic mean, *i.e.*, $E[G_n] \le E[A_n] = \mu_R$.*(The proof shows that this result again does not require independence of the* R_i.)

Remark 7 One dollar of wealth grows into $W_n = \prod_{i=1}^{n}(1+R_i)$ after n periods, and expected wealth is the initial wealth compounded for n periods at a uniform rate equal to the expected value of the estimator of the arithmetic mean, *i.e.*, $E[W_n] = (1+\mu_R)^n$.

Remark 8 One dollar to be received n periods in the future has a present value of $PVF_n = \prod_{i=1}^{n}\left(\frac{1}{(1+R_i)}\right)$, and expected present value is $E[PVF] = \mu^n_{(1/(1+R))}$.

Remark 9 *The e*xpected present value of one dollar exceeds the present value when the arithmetic mean is used in discounting; *i.e.*, $E(PVF_n) \ge \left(\frac{1}{1+\mu_R}\right)^n$.

Growth Rates from a Specific Observed Realization

Suppose we think of $R_1, R_2, \ldots, R_n$ as a specific realization of n values of R, then the following remarks (proved in the Appendix A) apply without taking expectations. *Remarks 10*-14 refer to simple algebraic calculations that are more mathematical than statistical in nature.

Remark 10 The geometric mean is less than or equal to the arithmetic mean, *i.e.*, $G_n \le A_n$.

Remark 11 Since one dollar of wealth grows into $W_n = \prod_{i=1}^{n}(1+R_i)$ after n periods, then actual terminal wealth is the initial wealth compounded for n periods at a uniform rate equal to the geometric mean, *i.e.*, $W_n = (1+G_n)^n$.

Remark 12 Since the present value of one dollar to be received in n periods is $PVF_n = \frac{1}{\prod_{i=1}^{n}(1+R_i)}$, then the actual present value may be calculated with the geometric mean. *i.e.*, $PVF_n = \frac{1}{(1+G_n)^n}$

Remark 13 If one dollar of initial wealth is compounded for n periods at the uniform rate A_n, then the compounded value is equal to or exceeds actual terminal wealth, *i.e.*, $(1+A_n)^n \geq W_n$. The equality sign holds if, and only if, $R_1 = R_2 = \ldots = R_n$.

Remark 14 If one dollar of future wealth is discounted for n periods at the uniform rate A_n, then the present value is equal to or less than actual present value, *i.e.*, $\frac{1}{(1+A_n)^n} \leq PVF_n$. The equality sign holds if, and only if, $R_1 = R_2 = \ldots = R_n$.

Notice that *Remarks 2, 6*, and *10* are intuitively consistent with each other in that

$G \leq A$	*Remark 2,*
$E[G_n] \leq E[A_n]$	*Remark 6,*
$G_n \leq A_n$	*Remark 10.*

That is, whether the geometric and arithmetic means are viewed in the context of an exact probability distribution, the expected values of estimators, or simple algebraic calculations from a particular realization of growth rates, the geometric mean is less than or equal to the arithmetic mean.

Remarks 3 and *7* also are intuitively consistent in that

$E(W) = (1+\mu_R)^n = (1+A)^n$	*Remark 3*
$E[W_n] = (1+\mu_R)^n = [1+E(A_n)]^n$	*Remark 7.*

These remarks say that it is the arithmetic mean that leads to the expected level of wealth when we work with the exact probability distribution of returns or the expected values of estimators.

Remarks 4 and *8* also are intuitively consistent in that

$E(PVF) = \mu^n_{(1/(1+R))}$	*Remark 4*
$E[PVF] = \mu^n_{(1/(1+R))}$	*Remark 8*

These remarks say that the mean of the $\frac{1}{(1+R)}$ random variable leads to the expected present value when we work with the exact probability distribution of returns or the expected values of estimators.

However, when we deal with a particular realization of returns, that is when

$$W_n = (1+G_n)^n \qquad \textit{Remark 11}$$

$$PVF_n = \frac{1}{(1+G_n)^n} \qquad \textit{Remark 12.}$$

The geometric mean gives the actual wealth level and present value, and the arithmetic mean leads to an over estimate of actual wealth and an underestimate of present value (see *Remark 13 and Remark 14*).

III. An Example Dealing with Expected Wealth

Roger Ibbotson (2002) has proffered an example to show that the arithmetic mean, when compounded over multiple periods, results in the expected level of wealth. In Ibbotson's example, R is a random variable that takes on only two values: $r_1 = .30$ and $r_2 = -.10$; both outcomes occur with equal probability $p_1 = p_2 = .50$. The mean of R is $\mu_R = .5(.30) + .5(-.10) = .10$. From definitions (1) and (2), we have

(1′) $$A = .5(1+.30) + .5(1-.10) - 1 = .10\text{, and}$$

(2′) $$G = (1+.30)^{.5}(1-.10)^{.5} - 1 = .0817.$$

Ibbotson extends his example to a second period leading to an expected accumulation of one dollar of initial wealth into

(3′) $$W = (.5)(.5)(1+.30)(1+.30) + (.5)(.5)(1-.10)(1+.30) + (.5)(.5)(1+.30)(1-.10) + (.5)(.5)(1-.10)(1-.10) = 1.21.$$

These results are consistent with *Remarks 1, 2*, and *3* since $A = \mu_R = .10, G \leq A$, and $W = (1+\mu_R)^n = (1+.10)^2 = 1.21$.

Suppose we change the distribution of returns in period 2 but retain the distribution used by Ibbotson in period 1. For period 1, the random variable R_1 takes on the values $r_{11} = .30$ and $r_{12} = -.10$ with probabilities $p_{11} = p_{12} = .50$. In period 2, let the random variable R_2 takes on values $r_{21} = .40$ and $r_{22} = .10$ with probabilities $p_{21} = .80$ and $p_{22} = .20$. The arithmetic mean return would be calculated from a generalization of formula (1), which we write as

$$(7) \qquad A = .5\left[\sum_{i=1}^{2} p_{1i}(1+r_{1i}) + \sum_{i=1}^{2} p_{2i}(1+r_{2i})\right] - 1$$

$$= (.5)[(.5)(1+.30) + (.5)(1-.10) + (.8)(1+.40) + (.2)(1+.10)] - 1 = .220.$$

Using formula (3), expected wealth after two periods would be

$$W = (.5)(.8)(1+.30)(1+.40) + (.5)(.2)(1+.30)(1+.10) + (.5)(.8)(1-.10)(1+.40) + (.5)(.2)(1-.10)(1+.10) = 1.474.$$

It is no longer true that one dollar compounded at the rate A equals expected wealth, since in the foregoing example $(1+.220)^2 = 1.488 > 1.474$.

Ibbotson (2002) says "[t]he arithmetic mean is the rate of return which, when compounded over multiple periods, gives the mean on the probability distribution of ending wealth value." In response to a criticism by Allyn Joyce (1995) that Ibbotson's example is flawed because it only contains two periods, Paul Kaplan (1995) shows that the arithmetic mean return, when compounded for 20 periods yields the correct value of expected wealth. However, neither Ibbotson nor Kaplan states the conditions under which his conclusion holds. As delineated in the Appendix A, *Remark 3* and *Remark* 7 are correct if the mean of R is unchanged during all n periods and returns are independent.[3]

From *Remark 11*, we know that it is the geometric mean (when compounded over multiple time periods) that gives the actual ending wealth value, whereas the arithmetic mean results in a wealth value in excess of its actual value (*Remark 13*). For example, Ibbotson (2007) shows a one dollar investment in large company stocks at the beginning of 1926 growing in value to $3,077.76 at the end of 2006. The arithmetic mean and geometric means [from formulae (5) and (6)] are $A_n = .1234$ and $G_n = .1042$, respectively. One dollar invested in 1926 compounded at the geometric mean of 10.42% grows into the actual observed wealth value of $3,077.76 by the end on 2006. However, one dollar compounded at the arithmetic mean of 12.34% would have grown into $12,396.18 by the end of 2006 – an amount approximately four times larger than its actual value. On these grounds we might ask whether $A_n = .1234$ is seriously deficient as an estimator of the arithmetic mean. Is A_n calculated from a small sample? Is A_n corrupted because the underlying mean return varies over time? Is A_n adversely affected because returns are highly correlated? The answer seems to be "no" to every question. Annual returns on large company stocks are exhibited graphically in Figure 1. A_n is calculated from eighty-one observations – a period covering virtually the entire modern history of the stock market. In addition, returns seem to be independent; the regression of the return in one

[3] There are two other papers dealing with Ibbotson's example. One paper is by George Cassiere (1996) and another paper by Joyce (1996) in which he responds to Kaplan. The Cassiere paper discusses a constant mean and independent returns.

year on the previous year's return is $\hat{R}_{Year} = .1200 + .0291R_{Year-1}$. The t values are 4.50 and .26 for the intercept and slope terms, respectively, and $R^2 = .0008$, with the correlation coefficient between returns in adjacent years being approximately the same as the slope coefficient .0291.[4] Beyond the visual impression from Figure 1 that the mean is stationary, the regression of returns on a linear time trend is $\hat{R}_{Year} = -.3420 + .000237(Year)$, with $Year$ = 1926,....,2006. There seems to be little time trend in returns. The t values are -.18 and .25 for the intercept and slope terms, respectively, $R^2 = .0008$, and the correlation coefficient between time and returns is .0277. However, the varying returns illustrated in Figure 1 guarantee that the strict inequality $(1 + A_n)^n > W_n$ version of *Remark 10* holds when we look at the specific realization of stock returns from 1926 to 2006.

To emphasize the role of the arithmetic mean in wealth accumulation, we offer the following example related to stock returns. we use the following facts as discussed above: the arithmetic mean is $A_n = .1234$, the geometric mean is $G_n = .1042$ for the period 1926-2006, and one dollar invested in 1926 grew to \$3,077.76 $= (1 + .1042)^{81} = (1 + G_n)^{81}$ at the end of 2006. Suppose we make the following assumption: annual rates of return on an investment will be $r_1 = .18$ with probability $p_1 = .80$ and $r_2 = -\ .1030$ with probability $p_2 = .20$. We further assume that this distribution of returns is unchanged for the next 81 years and that returns are independent. Then, the expected return is $A = \sum_{i=1}^{m} p_i(1 + r_i) - 1 = .8(1+.18) + .2(1-.1030) - 1 = .1234$, just as the arithmetic mean was in the stock market for 1926-2006. Now, consider the following question. What is the expected value of a one dollar investment 81 years from now? *Remark 3* tells us the answer: \$12,396.18 $= (1 + .1234)^{81} = (1 + A_n)^{81}$. This answer is illustrated in Table 1 which consists of Columns A, B, C, and D. Column A is the number of times the annual return is .18 in the next 81 years. Of course, 81 minus the number in Column A is the number of times the return is –.1030 in the next 81 years. Column B is the probability associated with Column A. The sum of the probabilities in B is 1.000. Column C measures the accumulated value of an investment of one dollar, given the entry in Column A. Column D is the product of Column B and Column C. The sum of Column D

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[4] The estimated intercept very closely approximates the arithmetic mean return, and the small and statistically insignificant slope term indicates that the return in any year is uncorrelated with the previous year's return. Finally, we note that $P_{t+1} = \delta + P_t + \varepsilon_{t+1}$ is the random walk model (with drift δ) for stock market prices. This is not quite our independent, and identically distributed random variable assumption of returns with mean μ. We assume $R_{t+1} = \mu + v_{t+1}$, with $R_{t+1} = \dfrac{P_{t+1} - P_t}{P_t}$. Our model implies

$P_{t+1} = P_t + \mu P_t + P_t \upsilon_{t+1}$: we need $\mu P_t \approx \delta$ and $P_t \upsilon_{t+1} \approx \varepsilon_{t+1}$. To the extent that these approximate equalities hold, our independent, and identically distributed, random variable assumption is consistent with the classic random walk model.

is the expected value of wealth accumulation after 81 years which is \$12,396.18 as shown by the lower right hand corner of Table 1. Complete details are presented in Table 1, but *Remark 3* gives us the short cut answer $\$12{,}396.18 = (1+.1234)^{81} = (1+A_n)^{81}$. Although the mean of the assumed binomial distribution is 64.8 = (.8)(81) and the median and mode are 65 up markets, various up and down market realizations could occur. Suppose 67 up markets occur (slightly more than the expected number), Table 1 shows accumulated wealth of \$14,295.15 which exceeds expected wealth of \$12, 396.18. Table 1 also shows that 66 or fewer up market realizations imply less than expected wealth. However, it is the arithmetic mean that gives us the correct expected value calculated in Table 1.

IV. Expected Present Value Biases Caused By Random Interest Rates, the Geometric Mean/Arithmetic Mean and Other Estimator Choices

Consider the case where we know that a certain future wage payment, FW_n will occur *n* periods into the future, and we wish to discount it to present value.[5] The funding vehicle earns returns R_i for periods *i* years into the future, $i = 1,2,\dots n$. FW_n might be the result of a union contract which has already been negotiated. Since \$1 today will be worth $W_n = (1+R_1)(1+R_2)\cdots(1+R_n)$ in *n* years, the present value random variable

$$(8)\ PV(FW_n; R_1, R_2, \cdots, R_n) = \frac{FW_n}{W_n} = \frac{FW_n}{(1+R_1)(1+R_2)\cdots(1+R_n)}$$

is the object of interest to forensic economists, and we would like a sensible statistical estimator of this. By estimator we mean that we will need to substitute values for the variables R_i into (8). As written, if the period of time is *n* years, then the R_i are one year rates, occurring $0,1,\dots n$-1 years into the future.

Define $R^{(n)} = \left((1+R_1)(1+R_2)\cdots(1+R_n)\right)^{1/n} - 1$ as the n-year geometric mean of these random variables. Its construction entails $(1+R^{(n)})^n = (1+R_1)(1+R_2)\cdots(1+R_n)$, so that in a sense, $R^{(n)}$ is a single sufficient statistic for the entire individual R_i. Of course, if one must estimate returns in not just period *n* but periods 1, 2, ..., *n*-1 as well, there is no data compression saving – *n* possibly different one period returns are involved. In general the problem involves estimating a central tendency measure of $R^{(n)}$ by some historical average, using data observed over the last *m* years.

Forensic economists use a variety of methods to address this problem. We list a few.

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[5] The more usual case we would assume that $FW_n = (1+G_1)(1+G_2)\cdots(1+G_n)$ where the G_i are random variables depicting the future growth rates of wages. These are likely *not* independent over time.

The Wall Street Journal (or Bloomberg) Method. Look up the *n* year yield on a Treasury strip and call this $R_0^{(n)}$. In the general framework, we can think of this method as setting *m*=0 years of historical data. Employing the previous result, where the left hand side is taken as the observed datum, $R^{(n)} = \left((1+R_1)(1+R_2)\cdots(1+R_n)\right)^{1/n} - 1$, so that we may use the fact that $(1+R^{(n)})^n = (1+R_1)(1+R_2)\cdots(1+R_n)$ in (8) and write

$$PV(FW_n;R_1,R_2,\cdots,R_n) = PV(FW_n;R^{(n)},R^{(n)},\cdots,R^{(n)}) = \frac{FW_n}{(1+R^{(n)})^n}$$

This can only be done when *n* is less than the longest Treasury, 30 years or so. Note that in effect this method assumes that the 1 year yields will all be equal to a common value, and that value is today's observable yield to maturity. Note further that, since $R_0^{(n)}$ locks in $R^{(n)}$ in today, this method avoids any uncertainty in future outcomes R_i. Put differently, letting R_i^e indicate the expected value now based on current information of R_i, the market expectations theory of the term structure of interest rates would yield $(1+R^{(n)})^n = (1+R_1)(1+R_2^e)\cdots(1+R_n^e)$.

Historical Average Method. Another approach is selected by forensic economists who do not see their assignment as constrained by today's interest rates. If they believe that today's rates are historically high or historically low, they may use estimators for $R_2,R_3,\cdots,R_n$ reflecting mean reversion. In particular, estimating $R^{(n)}$ by some historical average over the last *m*>1 years may capture this idea. Choice of *m* and choice of average are at issue. Popular values of *m* to be inserted into the chosen average include:
(a) 5 years, (b) 10 years, (c) 20 years, (d) 40-60 years, (e) a value of *m* chosen to rationalize an *a priori* rate, *e.g.* 2% or 3%.

Popular forms of averaging include the *GM* (geometric mean) and the *AM* (arithmetic mean):

$$GM(R_{-1},R_{-2},...,R_{-m}) \equiv \left((1+R_{-1})(1+R_{-2})\cdots(1+R_{-m})\right)^{1/m} - 1$$

$$AM(R_{-1},R_{-2},...,R_{-m}) = \frac{1}{m}\sum_{i=1}^{i=m} R_{-i} = \left(\frac{1}{m}\sum_{i=1}^{i=m}(1+R_{-i})\right) - 1$$

Although we have not seen anyone propose the harmonic mean *HM*

$$HM(R_{-1},R_{-2},...,R_{-m}) = \left(\frac{m}{\sum_{i=1}^{i=m}\frac{1}{(1+R_{-i})}}\right) - 1,$$

it may be a possible average of interest.[6]

Returning to our original problem, consider

$E[PV(FW_n;R_1,R_2,\cdots,R_n)]=E\left(\frac{FW_n}{(1+R_1)(1+R_2)\cdots(1+R_n)}\right)$. If the returns are independent identically distributed over time with mean $E(R_i)=\mu$, we have

$E[PV(FW_n;R_1,R_2,\cdots,R_n)]=FW_nE\left(\frac{1}{(1+R_1)}\right)E\left(\frac{1}{(1+R_2)}\right)\cdots E\left(\frac{1}{(1+R_n)}\right)$ and, since

$f(R_i)=\frac{1}{(1+R_i)}$ is a convex function of R_i, $E\left(\frac{1}{(1+R_i}\right)>\left(\frac{1}{(1+E(R_i)}\right)$, for all *i*, and

$E\big(PV(FW_n;R_1,R_2,\cdots,R_n)\big)>FW_n\left(\frac{1}{(1+\mu)}\right)^n$ (see *Re0mark 9*). In other words, if we choose a good estimator for μ such as the sample mean $AM=AM(R_{-1},R_{-2},...,R_{-m})$ and insert it into $\frac{FW_n}{(1+R_1)(1+R_2)\cdots(1+R_n)}$, the resulting $\frac{FW_n}{(1+AM)(1+AM)\cdots(1+AM)}$ will tend to underestimate the expected present value.[7] On the other hand, if we choose an estimator μ^* of μ which is downward biased, then $FW_n\left(\frac{1}{(1+\mu^*)}\right)^n$ will tend to exceed $FW_n\left(\frac{1}{(1+\mu)}\right)^n$ and therefore move in the direction of $\frac{FW_n}{(1+R_1)(1+R_2)\cdots(1+R_n)}$. Since $GM(R_{-1},R_{-2},...,R_{-m})<AM(R_{-1},R_{-2},...,R_{-m})$ with probability one in any sample, $E\{GM(R_{-1},R_{-2},...,R_{-m})\}<E\{AM(R_{-1},R_{-2},...,R_{-m})\}$, so that one candidate for μ^* is $GM(R_{-1},R_{-2},...,R_{-m})$. In fact, since $HM(R_{-1},R_{-2},...,R_{-m})<GM(R_{-1},R_{-2},...,R_{-m})$, the harmonic mean is another candidate. Without further assessment, there is no reason to favor the *GM* over the *HM* or vice versa.

Without independence of the R_i, the ability to proceed beyond $E\left(\frac{FW_n}{(1+R_1)(1+R_2)\cdots(1+R_n)}\right)$ is lost. Additionally, assessing randomness in the wage growth rates and interest rates simultaneously involves consideration of

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[6] See Appendix B for a geometrical based proof for the relation between the arithmetic and geometric means for the case of two numbers. The figure shows the harmonic mean and root mean square as well.
[7] Ibbotson (2002) seems to come to a different conclusion; he says "[t]he arithmetic mean ... serves as the correct rate for ... discounting ... ."

$$E\left(\frac{(1+G_1)(1+G_2)\cdots(1+G_n)}{(1+R_1)(1+R_2)\cdots(1+R_n)}\right)=E\left(\frac{1}{(1+NDR_1)(1+NDR_2)\cdots(1+NDR_n)}\right)$$, and the

$\frac{1}{1+NDR_i}$ are likely to be non-independent from business cycle reasons affecting the G_i, even if the returns R_i are independent.

V. Conclusion

We have defined the arithmetic mean, geometric mean, accumulated wealth, and present value based on a probability distribution, random sample, and a realization of returns. We have asserted and proved 14 remarks about these concepts. A few remarks are of central importance and worth reiterating. *Remark 13* referring to a particular realization of returns, says that a dollar compounded at the arithmetic mean will exceed (or equal if the returns are all the same) actual terminal wealth. Rather, it is the geometric mean that will take us from initial wealth to the ending wealth when dealing with a specific realization of returns (*Remark 11*). However, *Remark 3* (or *Remark 7*), referring to rates known from a probability distribution or a sample, says that a dollar compounded at the arithmetic mean grows into the expected wealth level of wealth. This paper gives conditions under which the latter statement holds, *viz*., rates of return have constant mean and are independent over time. One might look at things in the following manner. When we observe a particular realization of some variable over a period of years, the geometric mean will take us from the initial value of the variable to its terminal value. Such a calculation is retrospective. The geometric mean takes us from the initial to ending value of a realization by its very construction; this property is the defining characteristic of the geometric mean. Since the geometric mean calculated from a realization is retrospective, it will differ from other realizations of the same variable; and it possesses no particularly desirable properties for future realizations. However, suppose we want to be prospective, and we are interested in the future and what is expected to happen after a period of years. Since many different realizations can occur, we want to say something about the future that can be evaluated on a probabilistic basis. Suppose we have reasons to believe that investment returns are random with a constant mean and are independent. Then, the expected value of the investment is best estimated by compounding forward using the arithmetic mean (*Remark 3* and *Remark 7*). When we change the focus from wealth accumulation to present value, discounting with the arithmetic mean leads to a present value that is too small (*Remark 9* and *Remark 14*). When we observe a particular realization of some variable over a period of years, the geometric mean will take us from the terminal value of the variable back to its initial value by *Remark 12* – the counterpart of *Remark 11* for wealth accumulation. Since expected present value exceeds present value computed with the arithmetic mean, use of the geometric mean in the present value calculation leads to a larger present value calculation and thus moves towards the expected present value, but it is unclear whether the resulting present value overshoots the expected present value without further study. Without independence, certain of our results break down, and the time series dependence should be exploited in forecasting the returns in the event. This paper has harvested the low fruit in the forest of random returns and random wage growth rates.

Table 1. Expected Value of Wealth Accumulation for 81 Years Assuming Returns and Probabilities: $r_1 = .18$ with $p_1 = .80$ and $r_2 = -.1030$ with $p_2 = .20$

A	B	C	D	A	B	C	D	A	B	C	D
0	2E-57	0.000	4E-61	28	8E-19	0.324	3E-19	56	0.0066	700.17	4.62
1	8E-55	0.000	2E-58	29	6E-18	0.426	2E-18	57	0.0116	921.07	10.66
2	1E-52	0.000	3E-56	30	4E-17	0.561	2E-17	58	0.0192	1211.67	23.22
3	1E-50	0.000	5E-54	31	3E-16	0.738	2E-16	59	0.0299	1593.94	47.63
4	1E-48	0.000	5E-52	32	2E-15	0.971	2E-15	60	0.0438	2096.82	91.89
5	6E-47	0.001	4E-50	33	1E-14	1.277	1E-14	61	0.0603	2758.36	166.47
6	3E-45	0.001	2E-48	34	5E-14	1.680	9E-14	62	0.0779	3628.61	282.56
7	1E-43	0.001	1E-46	35	3E-13	2.210	6E-13	63	0.0939	4773.43	448.41
8	5E-42	0.001	7E-45	36	1E-12	2.907	4E-12	64	0.1057	6279.43	663.62
9	2E-40	0.002	3E-43	37	7E-12	3.824	3E-11	65	0.1106	8260.56	913.27
10	5E-39	0.002	1E-41	38	3E-11	5.030	2E-10	66	0.1072	10866.74	1165.00
11	1E-37	0.003	4E-40	39	1E-10	6.617	1E-09	67	0.0960	14295.15	1372.44
12	3E-36	0.004	1E-38	40	6E-10	8.705	5E-09	68	0.0791	18805.21	1486.83
13	6E-35	0.005	3E-37	41	2E-09	11.451	3E-08	69	0.0596	24738.18	1474.03
14	1E-33	0.007	8E-36	42	9E-09	15.064	1E-07	70	0.0409	32542.98	1329.65
15	2E-32	0.009	2E-34	43	3E-08	19.817	7E-07	71	0.0253	42810.17	1083.98
16	3E-31	0.012	4E-33	44	1E-07	26.069	3E-06	72	0.0141	56316.61	792.21
17	5E-30	0.016	8E-32	45	4E-07	34.294	1E-05	73	0.0069	74084.28	513.94
18	8E-29	0.021	2E-30	46	1E-06	45.114	6E-05	74	0.0030	97457.58	292.36
19	1E-27	0.027	3E-29	47	4E-06	59.347	2E-04	75	0.0011	128205.07	143.58
20	1E-26	0.036	5E-28	48	1E-05	78.071	8E-04	76	0.0004	168653.27	59.65
21	1E-25	0.048	7E-27	49	3E-05	102.701	3E-03	77	0.0001	221862.72	20.38
22	2E-24	0.063	1E-25	50	7E-05	135.103	1E-02	78	1.90E-05	291859.54	5.50
23	2E-23	0.082	1E-24	51	2E-04	177.728	3E-02	79	2.90E-06	383940.09	1.10
24	2E-22	0.108	2E-23	52	4E-04	233.800	9E-02	80	2.90E-07	505071.69	0.14
25	1E-21	0.142	2E-22	53	9E-04	307.563	3E-01	81	1.40E-08	664419.84	0.01
26	1E-20	0.187	2E-21	54	0.0018	404.6	0.7322				
27	1E-19	0.246	2E-20	55	0.0036	532.25	1.8913		1		12396.18

Source:

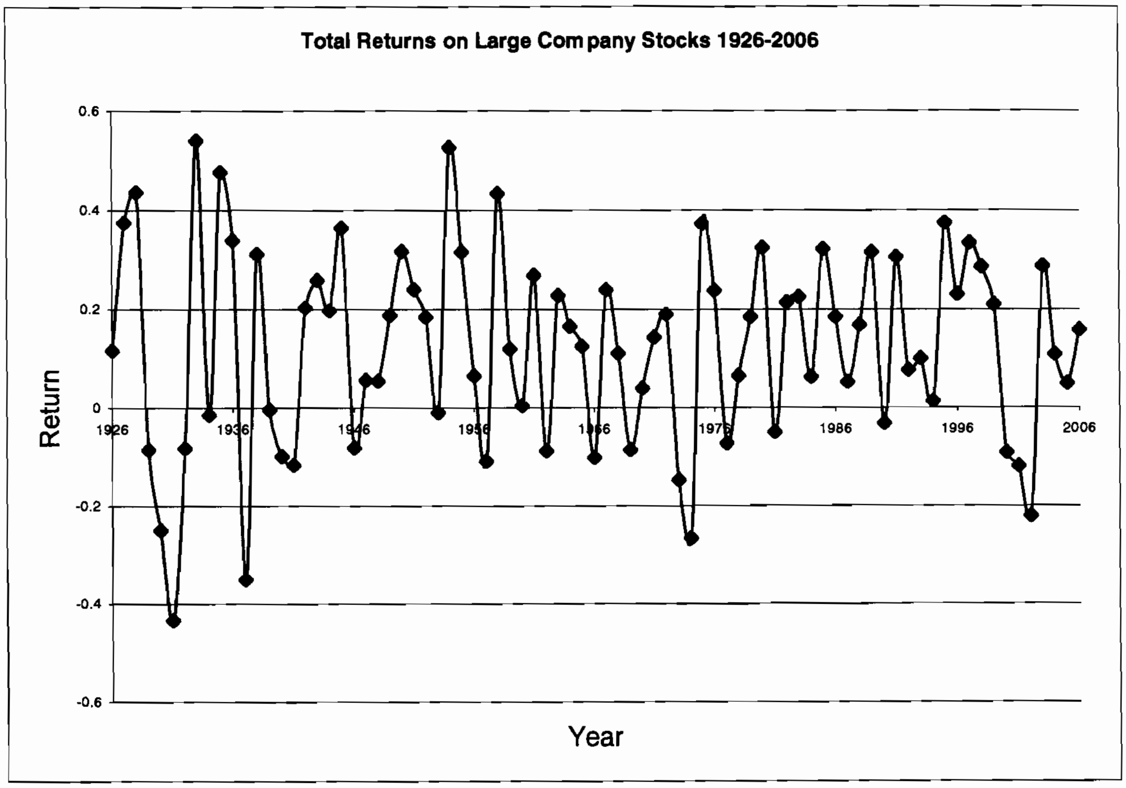
Column A shows the number of times the annual return is .18 in the next 81 years.

Column B contains the probability associated with Column A. Using x to denote the number in Column A, then the entry in Column B is $\frac{81!}{x!(81-x)!}(.8^x)(.2^{81-x})$. Some entries are in scientific notation where, for example, 3E-31 means 3.0×10^{-31}.

Column C measures the accumulated value of an investment of one dollar, given the entry in Column A. Each entry in Column C is $(1+.18)^x(1-.1030)^{81-x}$.

Column D is Column B x Column C.

Figure 1. Total Returns on Large Company Stocks 1926-2006



Appendix A: Proofs of Remarks

Proof of Remark 1

$$\begin{aligned} A &= \sum_{i=1}^{m} p_i(1+r_i) - 1 \\ &= \sum_{i=1}^{m} p_i + \sum_{i=1}^{m} p_i r_i - 1 \\ &= 1 + \mu_R - 1 \\ &= \mu_R = E[R] \end{aligned}$$

Proof of Remark 2

$G \le A$ if $G+1 \le A+1$, *i.e.*, if

$$\prod_{i=1}^{m} (1+r_i)^{p_i} \le \sum_{i=1}^{m} p_i(1+r_i)$$

.

We know that

$$\begin{aligned} \log\left[\prod_{i=1}^{m}(1+r_i)^{p_i}\right] &= \sum_{i=1}^{m} p_i \log(1+r_i) \\ &= E[\log(1+R)] \\ &\le \log E[1+R] \quad \text{(See Note at end of proof.)} \\ &\le \log \sum_{i=1}^{m} p_i(1+r_i). \end{aligned}$$

Since $\log\left[\prod_{i=1}^{m}(1+r_i)^{p_i}\right] \le \log \sum_{i=1}^{m} p_i(1+r_i)$ and since the log function is increasing, then

$$\prod_{i=1}^{m} (1+r_i)^{p_i} \le \sum_{i=1}^{m} p_i(1+r_i).$$

Note: This step is valid by Jensen's Inequality since the log function is concave.

Proof of Remark 3

$$E(W) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} p_{i_1} p_{i_2}\cdots p_{i_n}(1+r_{1i_1})(1+r_{2i_2})\cdots(1+r_{ni_n})$$

$$= \sum_{i_1=1}^{m} p_{i_1}(1+r_{1i_1}) \sum_{i_2=1}^{m} p_{i_2}(1+r_{2i_2}) \cdots \sum_{i_n=1}^{m} p_{i_n}(1+r_{ni_n})$$

$$= (1+\mu_R)(1+\mu_R)\cdots(1+\mu_R)$$
$$= (1+\mu_R)^n$$

Proof of Remark 4

$$E(PVF) = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} p_{i_1}p_{i_2}\cdots p_{i_n}\frac{1}{(1+r_{1i_1})(1+r_{2i_2})\cdots(1+r_{ni_n})}$$

$$= \sum_{i_1=1}^{m} p_{i_1}\frac{1}{(1+r_{1i_1})}\sum_{i_2=1}^{m} p_{i_2}\frac{1}{(1+r_{2i_2})}\cdots\sum_{i_n=1}^{m} p_{i_n}\frac{1}{(1+r_{ni_n})}$$

$$= E\left(\frac{1}{(1+r_{1i_1})}\right)E\left(\frac{1}{(1+r_{2i_2})}\right)E\left(\frac{1}{(1+r_{ni_n})}\right)$$

$$= \mu_{(1/(1+R))}\mu_{(1/(1+R))}\cdots\mu_{(1/(1+R))}$$

$$= \mu^n_{(1/(1+R))}$$

Proof of Remark 5

$$E[A_n] = E\left[\frac{1}{n}\sum_{i=1}^{n}(1+R_i)-1\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}E(1+R_i)-1$$
$$= \frac{1}{n}\sum_{i=1}^{n}(1+\mu_R)-1$$
$$= \frac{1}{n}(n+n\mu_R)-1$$
$$= \mu_R = E[R]$$

Observation and Lemma for Remark 6

Before proving the next remark, we need an observation and a lemma. The observation is that defining $x_i \equiv 1+R_i$ with $R_i \ge -1$ says that results known about functions defined on $x_i \ge 0$ permits us to make statements involving wealth accumulation factors $1+R_i$.

Lemma. $f(x_1,x_2,\cdots x_n) = \left[x_1x_2\cdots x_n\right]^{\frac{1}{n}}$ is a concave function of its arguments.

Proof. Since $f(x_1, x_2, \cdots x_n)$ is clearly twice continuously differentiable, it is necessary and sufficient to show that its second derivative matrix or Hessian $H \equiv \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$ is negative semi-definite, i.e. for every n-vector **v**, $v^T H v \leq 0$. We compute first

$\frac{\partial f}{\partial x_i} = \frac{1}{n} x_i^{\frac{1}{n}-1} \prod_{j \neq i} x_j^{\frac{1}{n}} = x \frac{f}{n x_i}$. Computing H, off the diagonal, for $i \neq j$,

$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{1}{n} x_i^{\frac{1}{n}-1} \frac{1}{n} x_j^{\frac{1}{n}-1} \prod_{\substack{k \neq i \\ k \neq j}} x_k^{\frac{1}{n}} = \frac{f}{n^2 x_i x_j}$ while on the diagonal

$\frac{\partial^2 f}{\partial x_i^2} = \frac{1}{n}\left(\frac{1}{n} - 1\right) x_i^{\frac{1}{n}-2} \prod_{k \neq i} x_k^{\frac{1}{n}} = \frac{1}{n}\left(\frac{1}{n} - 1\right) \frac{f}{x_i^2} = \frac{1}{n^2} \frac{f}{x_i^2} - \frac{1}{n} \frac{f}{x_i x_i}$. Thus

$H = \frac{-f}{n} dg\left\langle \frac{1}{x_i^2} \right\rangle + \frac{f}{n^2} \begin{pmatrix} \frac{1}{x_1} \\ \vdots \\ \frac{1}{x_n} \end{pmatrix} \begin{pmatrix} \frac{1}{x_1} & \cdots & \frac{1}{x_n} \end{pmatrix}$, where in the first term dg means a diagonal matrix with the indicated element on the diagonal. The computation proceeds with

$v^T H v = -\frac{f}{n} \sum_{i=1}^{i=n} \frac{v_i^2}{x_i^2} + \frac{f}{n^2} \left(\sum_{i=1}^{i=n} \frac{v_i}{x_i} \right)^2$. Multiply by n and divide by f and note that $v^T H v \leq 0$ if and only if $\frac{1}{n} \left(\sum_{i=1}^{i=n} \frac{v_i}{x_i} \right)^2 \leq \sum_{i=1}^{i=n} \frac{v_i^2}{x_i^2}$. But the Cauchy inequality says

$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$; take $a_i = \frac{v_i}{x_i}$ and $b_i = \frac{1}{\sqrt{n}}$ so $a_i b_i = \frac{v_i}{x_i \sqrt{n}}$,

$\left(\sum a_i b_i\right)^2 = \frac{1}{n}\left(\sum \frac{v_i}{x_i}\right)^2$ and $\sum a_i^2 = \sum \frac{v_i^2}{x_i^2}$, $\sum b_i^2 = \sum \frac{1}{n} = 1$ so the result follows.

Proof of Remark 6

$$E[G_n] = E\left[\left[\prod_{i=1}^{n} (1 + R_i)\right]^{\frac{1}{n}}\right] - 1$$

$$= E\left[\prod_{i=1}^{n} (1 + R_i)^{\frac{1}{n}}\right] - 1$$

$$\leq \left[\prod_{i=1}^{n} E(1+R_i)^{\frac{1}{n}}\right] - 1 \qquad \text{(See Note at end of proof.)}$$

$$\leq \left[\prod_{i=1}^{n} (1+\mu_R)^{\frac{1}{n}}\right] - 1$$

$$\leq \left[(1+\mu_R)^{\frac{1}{n}}\right]^n - 1$$

$$\leq 1 + \mu_R - 1$$

$$\leq \mu_R = E[A_n]$$

Note: This step is valid by Jensen's Inequality since $\prod_{i=1}^{n}(1+R_i)^{\frac{1}{n}}$ is a concave function by the lemma. In fact, from *Remark 10* below, since for any realization *s* with probability $p(s)$, $G_n(s) \leq A_n(s)$, multiplying by $p(s)$ and summing produces the result without any appeal to independence.

Proof of Remark 7

$$E[W_n] = E\left[\prod_{i=1}^{n}(1+R_i)\right]$$

$$= \left[\prod_{i=1}^{n} E(1+R_i)\right] \quad \text{by independence of a random sample}$$

$$= \left[\prod_{i=1}^{n}(1+\mu_R)\right]$$

$$= (1+\mu_R)^n$$

Proof of Remark 8

$$E[PVF] = E\left[\prod_{i=1}^{n}\left(\frac{1}{(1+R_i)}\right)\right]$$

$$= \left[\prod_{i=1}^{n} E\left(\frac{1}{(1+R_i)}\right)\right] \quad \text{by independence of a random sample}$$

$$= \left[\prod_{i=1}^{n} \mu_{(1/(1+R))}\right]$$

$$= \mu^n_{(1/(1+R))}$$

Proof of Remark 9

$$E[PVF] = E\left[\prod_{i=1}^{n}\left(\frac{1}{(1+R_i)}\right)\right]$$

$$= \left[\prod_{i=1}^{n} E\left(\frac{1}{(1+R_i)}\right)\right] \quad \text{by independence of a random sample}$$

$$\geq \left[\prod_{i=1}^{n}\left(\frac{1}{1+E(R_i)}\right)\right] \quad \text{by Jensen's Inequality since } f(R_i) = \frac{1}{1+R_i} \text{ is convex}$$

$$= \left[\prod_{i=1}^{n}\left(\frac{1}{1+\mu_R}\right)\right]$$

$$= \left(\frac{1}{1+\mu_R}\right)^n$$

Note: f is convex since $f''(R_i) = 2(1+R_i)^{-3} > 0$

Proof of Remark 10

The proofs of *Remark 2* and *Remark 5* involve random variables. Here we assume that $R_1, R_2, ..., R_n$ are real numbers greater than or equal to –1. First, we note that $A_n \geq G_n$ if $A_n + 1 \geq G_n + 1$. Then, from Jensen's Inequality and the fact the natural log function is concave,

$$\ln\left(\sum_{i=1}^{n}(1+R_i)/n\right) \geq (1/n)\sum_{i=}^{n}\ln(1+R_i) = \ln\left(\prod_{i=1}^{n}(1+R_i)\right)^{1/n}.$$

Since the natural log function is an increasing function,

$$\left(\sum_{i=1}^{n}(1+R_i)/n\right) \geq \left(\prod_{i=1}^{n}(1+R_i)\right)^{1/n}$$

$$A_n + 1 \geq G_n + 1$$

$$A_n \geq G_n$$

$$G_n \leq A_n$$

There are many ways to prove the arithmetic-geometric mean inequality, among the most storied inequalities of all mathematics. The simplest proof from first principles notes that $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 < \left(\frac{a+b}{2}\right)^2$ unless $a=b$, in which case we have equality. If a and b are positive, we have the result for n=2. Now repeating this with $cd = \left(\frac{c+d}{2}\right)^2 - \left(\frac{c-d}{2}\right)^2 < \left(\frac{c+d}{2}\right)^2$ and multiplying gives $abcd \le \left(\frac{a+b}{2}\right)^2 \left(\frac{c+d}{2}\right)^2 \le \left(\frac{a+b+c+d}{4}\right)^4$ where the last equality follows from another application of the n=2 case. This proves the result for n=4. Proceeding upwards in powers of 2, the result follows for all $n = 2^m$. The intermediate values of n which are not powers of 2 may be filled in by using the result above for the higher n and adeptly choosing an arithmetic mean to extend the desired sequence in n to the next higher power of 2 – see Hardy, Littlewood and Polya (1934), p. 17.

Proof of Remark 11

From definition (6), we have

$$G_n = \left[\prod_{i=1}^{n}(1+R_i)\right]^{\frac{1}{n}} - 1.$$

$$G_n + 1 = \left[\prod_{i=1}^{n}(1+R_i)\right]^{\frac{1}{n}}$$

$$(G_n + 1)^n = \left[\prod_{i=1}^{n}(1+R_i)\right]$$

$$(G_n + 1)^n = W_n$$

Proof of Remark 12

In the last line of Remark 11, replace W_n with 1, then the present value of 1 is

$$PVF_n = \frac{1}{(1+G_n)^n}.$$

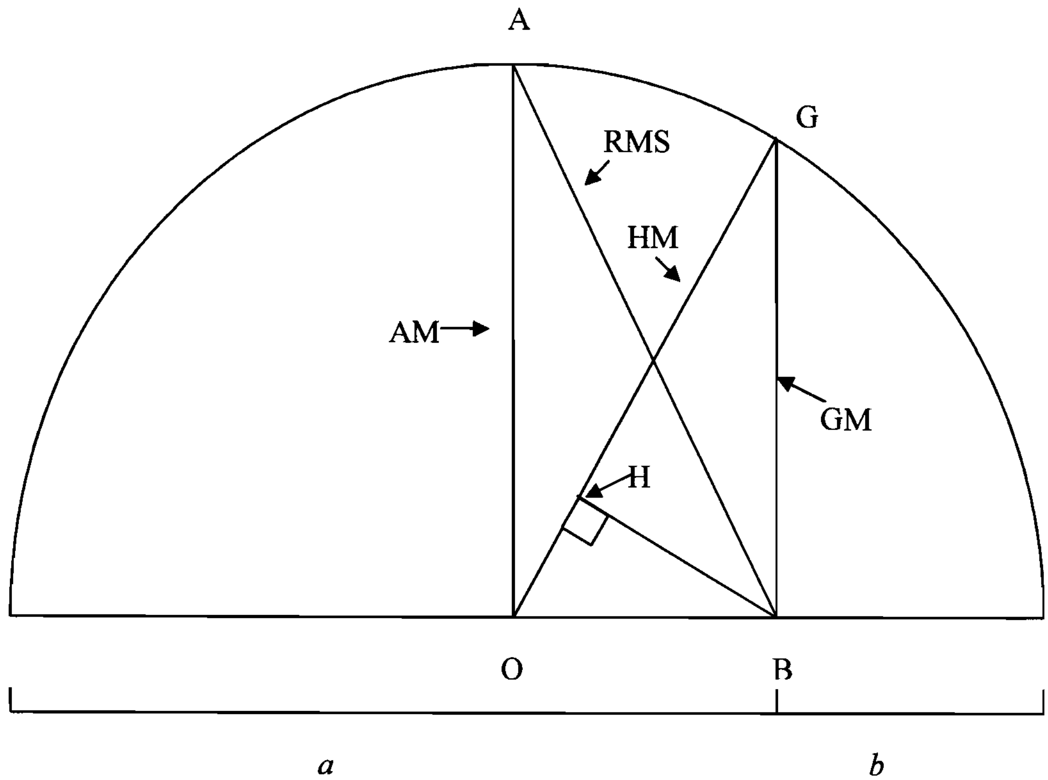
Proof of Remark 13

By combining *Remarks 10* and *11*, we have $(1+A_n)^n \ge W_n$.

Proof of Remark 14

By combining *Remarks 10* and *12*, we have $\frac{1}{(1+A_n)^n} \le PVF$.

Appendix B



Source: http://www.artofproblemsolving.com/Wiki/index.php/RMS-AM-GM-HM

The picture illustrates the following six-part inequality for $n = 2$ numbers:

$$\max(x_1, x_2, \cdots x_n) \geq RMS \equiv [(x_1^2 + x_2^2 + \cdots x_n^2)/n]^{.5} \geq AM \equiv (x_1 + x_2 + \cdots x_n)/n$$

$$\geq GM \equiv (x_1 x_2 \cdots x_n)^{\frac{1}{n}} \geq HM \equiv n/(x_1^{-1} + x_2^{-1} + \cdots + x_n^{-1}) \geq \min(x_1, x_2, \cdots x_n)$$

with equality if and only if the x_i are all equal. The picture shows two x_i values, $x_1 = a$ and $x_2 = b$. In the picture, the arithmetic mean AM is the distance OA, the geometric mean GM is the distance BG, the harmonic mean HM is the distance HG, and the root-mean square RMS is the distance BA. Extreme values are the maximum value *a* and the minimum value *b*. The arithmetic mean is $AM = (a+b)/2 = 2(OA)/2 = \text{OA}$. To find the geometric mean, we observe that $(\text{OB})^2 = [((a+b)/2) - b]^2 = [(a-b)/2]^2$. Also, $(\text{GB})^2 = (\text{OG})^2 - (\text{OB})^2 = [(a+b)/2]^2 - [(a-b)/2]^2 = ab = GM^2$. Therefore, GB = *GM.*

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