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Interchangeability of the median operator with the present value operator

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It is well known that the expected present value of a life annuity is smaller than the present value of an annuity certain with term equal to life expectancy. This result can be viewed as a consequence of the lack of interchangeability between the present value operator and the mathematical expectation operation. However, we prove that the median and present value operators can be interchanged; that is, the median of the present value of a life annuity equals the present value of an annuity certain whose term is the median additional years of life. At young ages and through late middle age, median additional years of life exceed life expectancy. Therefore, the median value of a life annuity exceeds an annuity certain paid to life expectancy which exceeds the expected present value of a life annuity - the fair price of a life annuity.

Keywords: annuity; median present value; expected value; life annuity

JEL Classification: G00; G19

I. Introduction

Modern understanding of life expectancy and median years of additional life dates to the work done by Christiaan Huygens (1669) and his brother Lodewijk. They drew a sharp distinction between these concepts but recognized the usefulness of both. Before the end of the seventeenth century, Jan de Witt (1671) and Edmond Halley (1693) provided the correct understanding of the expected present value of a life annuity although they approached life annuities differently. Expected present value is undoubtedly the correct concept upon which to price life annuities. However, a purchaser of a life annuity also might be interested in the median present value of the life annuity just as a person might be interested in median additional years of life. It is well known that a life annuity is worth less than an annuity certain paid to life expectancy. In this article, we consider the relationship between the median present value of a life annuity and the present value of an annuity certain paid to the median additional years of life. We show that what is not true for expectations is indeed true for medians. That is, the median operator and present value operator can be interchanged even though the expectation and present value operators cannot.

II. Notation

Let l_x denote the number of survivors in a population at exact age x, and YAL_x denote the years-of-additional-life random variable for that population. YAL_x takes on the value $t - 0.5$ with probability $(l_{x+t-1} - l_{x+t})/l_x$ if a person survives for $t - 1$ years beyond age x and dies before the completion of t years beyond age x, where $t = 1, 2, \ldots, 111 - x$ and 111 is taken to be the

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youngest age at which there are no survivors. Complete life expectancy at age x is defined as

$$\dot{e}_x \equiv E(YAL_x) = \sum_{t=1}^{111-x} (t-0.5)\left(\frac{l_{x+t-1}-l_{x+t}}{l_x}\right) \quad \text{for } x = 0, 1, \ldots, 110 \tag{1}$$

The present-value-of-an-annuity-certain random variable paying one at mid-year for each additional year of life and with positive discount rate r is defined as

$$PV(YAL_x) \equiv a_{\overline{YAL_x}|} \equiv \sum_{i=0.5}^{i=YAL_x} \frac{1}{(1+r)^i} = \frac{1}{r}\left[(1+r)^{0.5} - (1+r)^{-YAL_x}\right] \text{ for } r > 0 \tag{2}$$

When $YAL_x = t-0.5$, $PV(YAL_x) = a_{\overline{t-0.5}|}$ by construction from Equation 2; and both $YAL_x = t-0.5$ and $PV(YAL_x) = a_{\overline{t-0.5}|}$ occur with the same probability of $(l_{x+t-1} - l_{x+t})/l_x$.

Taking expectations in Equation 2 results in the following life annuity expression[1]:

$$a_x \equiv E[PV(YAL_x)] = E\left\{\sum_{i=0.5}^{i=YAL_x} \frac{1}{(1+r)^i}\right\} = \frac{1}{r}\left[(1+r)^{0.5} - E\left\{(1+r)^{-YAL_x}\right\}\right] \tag{3}$$

while inserting $\dot{e}_x = E(YAL_x)$ into the annuity-certain definition results in

$$a_{\overline{\dot{e}_x}|} \equiv \sum_{i=0.5}^{i=E(YAL_x)} \frac{1}{(1+r)^i} \equiv \frac{1}{r}\left[(1+r)^{0.5} - (1+r)^{-E(YAL_x)}\right] \tag{4}$$

The well-known actuarial inequality $a_x < a_{\overline{\dot{e}_x}|}$ arises when comparing Equation 3 with Equation 4.[2]

III. Median Annuity Theorem

The two most important descriptive measures of central tendency for additional years of life are

$\dot{e}_x \equiv E(YAL_x)$ – average years of additional life, or life expectancy; and

$\text{med}(YAL_x)$ – the median years of life, when 50% of a population has died.

An individual may contemplate purchasing an annuity, a random variable, paying the person while alive. With the purchase of such an annuity, the present value of the uncertain stream has two related measures of central tendency in which a purchaser may be interested:

$a_x \equiv E[PV(YAL_x)]$ – the expected (or average) present value of the annuity; and

$\text{med}[PV(YAL_x)]$ – the median present value, the result achieved by 50% of purchasers.

There also are two annuities certain, associated with each of the YAL_x characteristics:

$a_{\overline{\dot{e}_x}|}$ – the annuity certain to life expectancy; and

$a_{\overline{\text{med}(YAL_x)}|}$ – the annuity certain to the median years of life remaining.

We may now state the motivation for this article. Beyond the inequality $a_x < a_{\overline{\dot{e}_x}|}$, what is true theoretically and empirically about relationships among the foregoing concepts? Equation 3 differs from Equation 4 because *the present value PV operation and the mathematical expectation operation E cannot*

[1] The fundamental idea for this representation of a life annuity dates to Jan de Witt (1671) (Hendricks, 1852/1853; Haberman and Sibbett, 1995 for de Witt's report on life annuities, and Poitras, 2000). De Witt (1671) used a weighted average of annuities certain where the weights were mortality probabilities, thereby producing the expected value of the present value of the life annuity. Edmond Halley's (1693) representation of the life annuity dates to 1693 when he expressed the life annuity as $1/(1+r)^i$ multiplied by the probability of surviving at least i years and summed over i.

[2] George King (1878, 1887/1902) discussed an analogy between an annuity certain and a life annuity in 1878 and gave a rigorous proof in 1887 as did E. F. Spurgeon (1929). King's (1878, 1887/1902) proof uses the idea that survival probabilities occurring in the more distant future (and thereby most affected by discounting in a life annuity) are less heavily discounted in an annuity certain to life expectancy where all survival probabilities are lumped into life expectancy. Spurgeon's (1929) proof relies on the result that the arithmetic mean of positive quantities exceeds the geometric mean. Neither King (1878, 1887/1902) nor Spurgeon (1929) cited a previous proof. It is unclear whether King (1878, 1887/1902) produced the first rigorous proof, or whether a proof was given earlier (Kopf, 1925). Useful references include Steffensen (1919), Sarason (1960) and Olson (1975). This inequality has had a checkered and storied past, having been asserted as an equality in the early actuarial science literature. Indeed, some continue to hold this incorrect view in very modern times. For example, Hacking (1999) asserted 'The fair price for £100 for life must be the same as that for a terminal annuity for n years, where n is the expectation of life.'

be interchanged. It is natural to inquire about whether there is a corresponding inequality between the present value operation, which is the essence of an annuity, and the *med* operation. The chief result of this article is that the intuition about lack of interchangeability embodied in $a_x < a_{\overline{\mathring{e}_x}|}$ is wrong when it comes to the median and present value operations, that is, *med* and *PV do* interchange. It will be shown that $a_{\overline{\text{med}(YAL_x)}|} = \text{med}[PV(YAL_x)]$; an annuity certain evaluated at the median years of additional life equals the median of the present value random variable of additional years of life.

We need to establish a result for medians before proving the *Median Annuity Theorem* below. We start with the general definition of the median.

Definition 1: x_{med} is the median of the random variable X, also denoted med(X), if and only if it satisfies $P[X \leq x_{\text{med}}] \geq 0.5$ and $P[X \geq x_{\text{med}}] \geq 0.5$.

Lemma 2: *Let* $g(x)$ *be monotonic – either nondecreasing or nonincreasing. Then for any random variable* X, $\text{med}[g(X)] = g[\text{med}(X)]$.

Proof 2: Assume $g(x)$ is increasing. The events $[X \leq x_{\text{med}}]$ and $[g(X) \leq g(x_{\text{med}})]$ are the same events, and so have the same probability, which by definition equals or exceeds 0.5. The same is true of $[X \geq x_{\text{med}}]$ and $[g(X) \geq g(x_{\text{med}})]$. If $g(x)$ is decreasing, $[X \leq x_{\text{med}}]$ and $[g(X) \geq g(x_{\text{med}})]$ are the same events, and each with probability greater than, or equal to, 0.5; and the same statement holds for the opposite inequalities in the second part of the median definition. The lemma now follows by definition.

Median Annuity Theorem: $a_{\overline{\text{med}(YAL_x)}|} = \text{med}[PV(YAL_x)]$.

Proof: In Equation 2, $PV(YAL_x)$ is an increasing function of YAL_x. Applying Lemma 2 with $PV(\cdot)$ replacing $g(\cdot)$ and YAL_x replacing X gives the result

$$\text{med}[PV(YAL_x)] = PV[\text{med}(YAL_x)] \equiv a_{\overline{\text{med}(YAL_x)}|}$$

The result is illustrated with the closed form expression for $PV(\cdot)$ as follows:

$$\text{med}[PV(YAL_x)] \equiv \text{med}\left\{\sum_{i=0.5}^{i=YAL_x} \frac{1}{(1+r)^i}\right\} = \frac{1}{r}\left[(1+r)^{0.5} - (1+r)^{-\text{med}(YAL_x)}\right]$$

since medians may be interchanged with constants, and $-(1+r)^{-YAL_x}$ is monotonic. However,

$$a_{\overline{\text{med}(YAL_x)}|} \equiv \sum_{i=0.5}^{i=\text{med}(YAL_x)} \frac{1}{(1+r)^i} = \frac{1}{r}\left[(1+r)^{0.5} - (1+r)^{-\text{med}(YAL_x)}\right]$$

The right-hand sides are equal, illustrating the general result and proving it here in particular.[3] □

Consider Table 1. For young to late middle-aged men (approximately age $x < 70$),[4] we have

Table 1. Life expectancy, median years of life, life annuity, annuity certain to life expectancy and annuity certain to median years of life for men using 0.02 discount rate

Age x	$\mathring{e}_x$	$\text{med}(YAL_x)$	a_x ($)	$a_{\overline{\mathring{e}_x}\|}$ ($)	$a_{\overline{\text{med}(YAL_x)}\|} = \text{med}[PV(YAL_x)]$ ($)
0	75.1	78.5	38.4	39.2	39.9
10	65.8	69.5	36.2	36.9	37.9
20	56.1	59.5	33.2	34.0	35.1
30	46.9	49.5	29.9	30.7	31.7
40	37.6	39.5	25.9	26.8	27.6
50	28.8	30.5	21.5	22.2	23.2
60	20.7	21.5	16.7	17.3	17.8
70	13.6	13.5	11.9	12.3	12.2
80	7.8	7.5	7.4	7.7	7.4
90	4.1	3.5	4.3	4.4	3.8

Source: $\mathring{e}_x$ from Arias (2010, table 2).
Notes: $\mathring{e}_x$ reproduced with Equation 1. $\text{med}(YAL_x)$ computed with the definition $P[X \leq x_{\text{med}}] \geq 0.5$ and $P[X \geq x_{\text{med}}] \geq 0.5$. a_x computed with Equation 3. $a_{\overline{\mathring{e}_x}|}$ computed with Equation 4. $a_{\overline{\text{med}(YAL_x)}|}$ computed using Equation 2 evaluated at $\text{med}(YAL_x)$.

[3] A related result holds for the mode. That is, the annuity certain evaluated at the modal years of additional life equals the mode of the present value function of additional years of life; $a_{\overline{\text{mod}(YAL_x)}|} = \text{mod}[PV(YAL_x)]$.

[4] Survival data are from *United States Life Tables* 2006, table 2, Arias (2010).

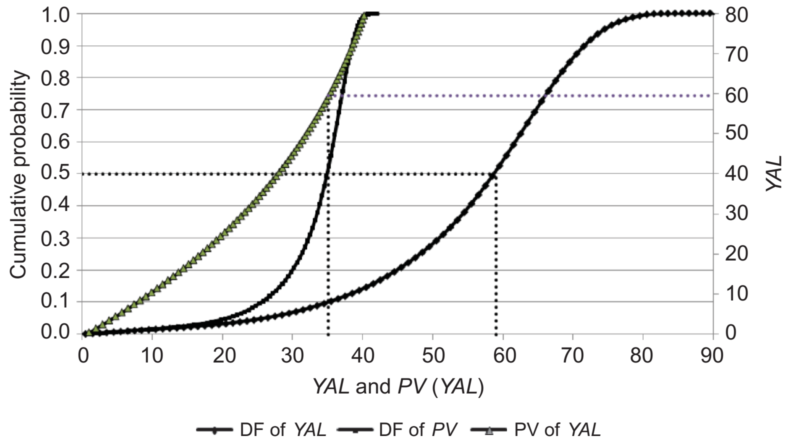

Fig. 1. Illustration of the *Median Annuity Theorem* for men age 20 and $r = 0.02$

$\dot{e}_x < \text{med}(YAL_x)$, but $\dot{e}_x > \text{med}(YAL_x)$ when the age inequality changes. Combining these empirical results with the above theorem, we have (for men age $x < 70$)

$$a_x = E[PV(YAL_x)] < a_{\overline{\dot{e}_x|}} < \text{med}[PV(YAL_x)] = a_{\overline{\text{med}(YAL_x)|}}.$$

For older men ($x \geq 70$)

$$a_{\overline{\dot{e}_x|}} > \text{med}[PV(YAL_x)] = a_{\overline{\text{med}(YAL_x)|}}$$

Figure 1 illustrates the *Median Annuity Theorem* for men aged 20 with $r = 0.02$. The cumulative Distribution Function (DF), the integral or indefinite sum of the probability mass function, is the natural representation with which to illustrate the determination of medians, since the median occurs where the horizontal line at 0.5 crosses the DF. The DF of YAL_{20} (the right-most function) and the DF of $PV(YAL_{20})$, the middle function, show $\text{med}(YAL_{20}) = 59.5$ and $\text{med}[PV(YAL_{20})] = \$35.1$, respectively. Starting with $\text{med}(YAL_{20}) = 59.5$ on the right-hand scale and moving to the present value of an annuity-certain function (left-most function), we see that $PV[\text{med}(YAL_{20})] = a_{\overline{\text{med}(YAL_{20})|}} = a_{\overline{59.5|}} = \$35.1$ as well. That is, the median of the present value equals the present value function evaluated at the median additional years of life – the *Medium Annuity Theorem*.

IV. Conclusion

The median operator can be interchanged with the annuity-certain operator, unlike the relationship between the expectation and annuity-certain operators. Therefore, the median of the present value of a life annuity has a value equal to an annuity certain with term equal to median additional years of life. For young to late middle-aged people, the median value of a life annuity exceeds the value of an annuity certain with life expectancy as its term, which always exceeds the expected value of a life annuity. The results for the median easily generalize for any cumulative probability characteristic like the interquartile range. The results for the median also hold for joint-life annuities and last-survivor annuities.

The fair price of a life annuity is its expected present value, and annuity contracts are correctly based on expected present value. However, a purchaser of a life annuity also might be interested in knowing its median present value. Christiaan Huygens (1669) identified the median additional years of life with the idea of 'wagering'. He said

> There are therefore two different choices for the expectation or the value of the future age of a person, and the age to which there is equal likelihood that he will survive or not survive. The first is in order to set [a] life pension, and the other for wagers.

In the same way, a person might be interested in both the expected present value and the median present value of a life annuity. The *Median Annuity Theorem* tells us that half of a population at risk will receive an annuity whose value is at least equal to an annuity certain whose term is fixed at median additional years of life. Said differently, more than half of the population will receive an annuity that exceeds a_x for $x < 70$. Consider a 20-year-old male in Table 1. Life expectancy is $\dot{e}_{20} = 56.1$ years and $\text{med}(YAL_{20}) = 59.5$

years. The fair price of a life annuity is $a_{20} = \$33.2$, but half of the population at risk will receive at least $a_{\overline{\text{med}(YAL_x)}|} = \$35.1$. That is, a 'Huygens' betting man' might be attracted to an annuity whose price is \$33.2 since the probability of receiving an annuity exceeding that fair purchase price is greater than 50%.

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