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The Relation between Two Present Value Formulae

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David Jones recently raised an interesting question on a forensic LISTSERV. He observed that a hand-held calculator returned $4.26284 for the present value of an ordinary annuity of $1 for 4.5 years when evaluated at a discount rate of .02. However, the present value of annual payments of $1 for 4.0 years is $3.80773; and the present value of a final payment of $.50 in 4.5 years is .45737 – a total of $4.26510. The difference between $4.26284 and $4.26510 is small, but the question is why the two present values differ.

In general notation, the Jones question could be phrased as follows: Let n be an integer number of years, i denotes the discount rate (assumed to be greater than zero and less than or equal to one), and let $0<\theta<1$ denote a fraction of a year and the amount of the payment made in the fractional year. In Jones's question, $n = 4$, $\theta = .5$, $n+\theta = 4.5$, and $i = .02$. A hand-held calculator computes

(1) $$(1/i)[1-(1+i)^{-(n+\theta)}]$$

as distinct from

(2) $$(1/i)[1-(1+i)^{-n}]+\frac{\theta}{(1+i)^{n+\theta}} \; .$$

This note investigates the relation between formulae (1) and (2).

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First, we observe,[1]

(3) $1+\theta i>(1+i)^{\theta}$.

From inequality (3), we have

(4a) $1+\theta i>(1+i)^{\theta}$ repeating (3)

(4b) $-(1+i)^{\theta}+\theta i>-1$ rearranging (4a)

(4c) $(1+i)^{n+\theta}-(1+i)^{\theta}+\theta i>(1+i)^{n+\theta}-1$ adding $(1+i)^{n+\theta}$ to both sides of (4b)

(4d) $(1+i)^{n+\theta}[1-(1+i)^{-n}]+\theta i>(1+i)^{n+\theta}[1-(1+i)^{-(n+\theta)}]$ regrouping (4c)

(4e) $(1/i)[1-(1+i)^{-n}]+\dfrac{\theta}{(1+i)^{n+\theta}}>(1/i)[1-(1+i)^{-(n+\theta)}]$

multiplying (4d) by $i^{-1}(1+i)^{-(n+\theta)}$

Since the left side of (4e) is formula (2) and the right side is formula (1), we have established that formula (2) exceeds formula (1) for $0<\theta<1$, $0<i\leq 1$, and positive integer values *n* as exemplified in the Jones question.

The difference between formulae (1) and (2) can be approximated by expanding formulae (1) and (2) with the general binomial theorem.[2]

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[1] This can be seen from the expansion of $(1+i)^{\theta}$ using the general binomial theorem which gives:
$(1+i)^{\theta}=1+\theta i+(1/2)\theta(\theta-1)i^{2}+(1/6)\theta(\theta-1)(\theta-2)i^{3}+\ldots$, noting that the third term in the expansion is negative and the fourth is positive but smaller than the absolute value of the third term. Successive pairs of terms follow the same pattern. Therefore, $(1+i)^{\theta}=1+\theta i+$ [a negative amount], and
$1+\theta i>(1+i)^{\theta}$.

[2] See Appendix 1 for this result.

(5) $$\frac{1}{2}\left[\frac{\theta(1-\theta)}{(1+i)^n}\right]i$$

For example, in the Jones formulation, (5) evaluates to

(5a) $$\frac{1}{2}\left[\frac{\theta(1-\theta)}{(1+i)^n}\right]i = \frac{1}{2}\left[\frac{.5(1-.5)}{(1+.02)^4}\right](.02) = .00231$$

where the actual difference is .00226 = 4.26510 – 4.26284.

The difference between formulae (1) and (2) is small (especially for net discount rates used in forensic work), but (2) does exceed (1). Formula (1) can be viewed as an extension of the formula for an annuity immediate but with a non-integer term rather than an integer term. One might think of it as equivalent to a level annuity paid at points in time $\frac{n+\theta}{n+1}, \frac{2(n+\theta)}{n+1}, \frac{3(n+\theta)}{n+1}, \ldots, \frac{(n+1)(n+\theta)}{n+1}$ [*i.e.*, at equal intervals of $(n+\theta)/(n+1)$]. We can then find the periodic payment that would be just sufficient to make the present value of payments equal to the value produced by formula (1). For example, in Jones's question, payments would be made at points in time

$$\frac{4+.5}{4+1}, \frac{2(4+.5)}{4+1}, \frac{3(4+.5)}{4+1}, \frac{4(4+.5)}{4+1}, \frac{(5)(4+.5)}{4+1},$$

which simplifies to .9, 1.8, 2.7, 3.6, 4.5 years into the future. A level annuity of $.899107 has a present value of $4.26284 as results from formula (1); and, in that sense, is equivalent to formula (1). Figure 1 is the time diagram for this annuity. Figures 2a and 2b show time diagrams using formula (2) and an annuity equivalent to (2), both of which equal $4.26510. Figure 2b shows slightly higher payments than Figure 1 at exactly the same points in time, resulting in a greater present value. This result is consistent with inequality (4e) which established that formula (2) produces a greater present value than formula (1).

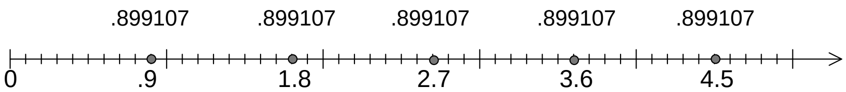


Figure 1. Present Value of Payments Equivalent to Formula 1

Present Value = $4.26284

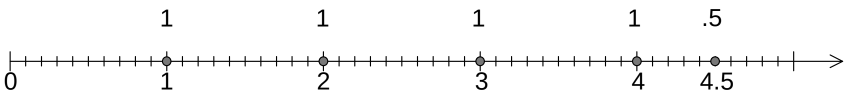


Figure 2a. Present Value of Payments Using Formula 2

Present Value = $4.26510

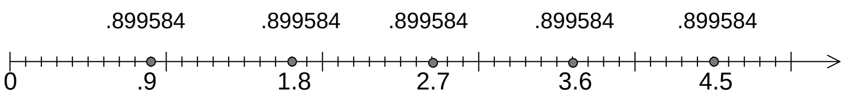


Figure 2b. Present Value of Payments Equivalent to Formula 2

Present Value = $4.26510

We can say more if we view the two objects being compared as functions of a real variable, *s*, defined on the positive real numbers. Using the usual notation [*s*] to indicate the greatest integer in *s*, $n = [s]$ and $s = n + \theta = [s] + \theta$, we have, for all positive *s*,

from (1), $PVSS(s;i) = (1/i)[1-(1+i)^{-s}]$

from (2) $PVEXACT(s;i) \equiv (1/i)[1-(1+i)^{-[s]}] + \dfrac{(s-[s])}{(1+i)^{s}}$.

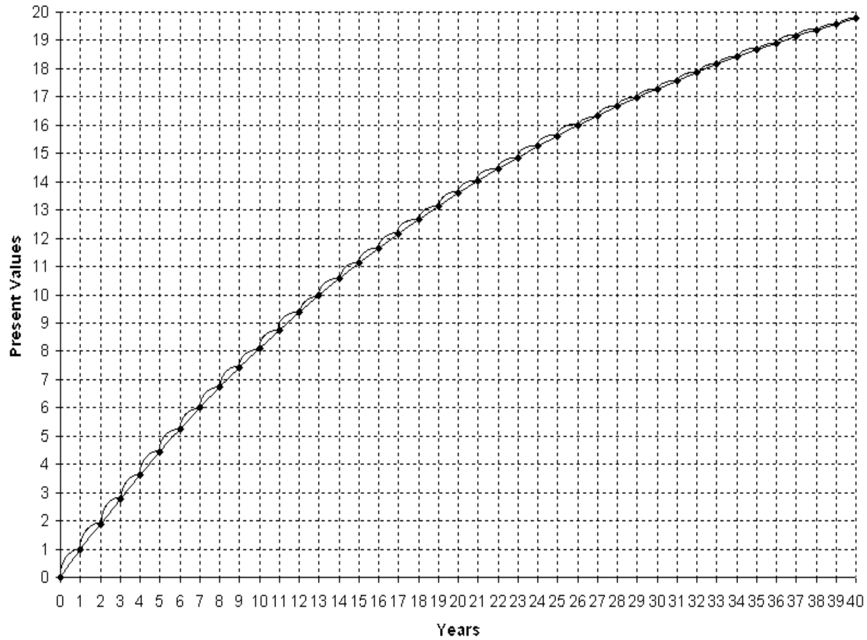
The notation *PVSS* is chosen to reflect the present value function embedded in commercial spreadsheets, in particular, in Microsoft Excel. In fact, the Excel function is $PV(i, nper, pmt, fv, type)$, with *nper* being the number of periods – our *s*. Additionally, *pmt*, the payment per period, is 1, *fv* (which can convert the spreadsheet to a future value calculation) is set to 0, and *type*, which governs whether the payments occur at the beginning or the end of periods, is set to 0 to reflect our end-of-period assumption. Thus $PVSS(s;i) \equiv PV(i, s, 1, 0, 0)$. In fact, the Microsoft help screen for *PV* discusses *nper* as if it were an integer – it is not even clear that Microsoft gave any thought to the problem being discussed here.

Now because Excel has the [*s*] built in as its *INT*(*s*) function, the forensic economist wishing to avoid the small error being discussed here can avoid it by creating his own user-defined function within Excel using our formula for *PVEXACT*. The earlier discussion showed that $PVEXACT(s;i) \geq PVSS(s;i)$, with equality on the integers and inequality off the integers.

Both $PVEXACT(s;i)$ and $PVSS(s;i)$ are continuous, monotonically increasing functions on $(0,\infty)$ which agree with $(1/i)[1-(1+i)^{-(s)}]$ when *s* takes on the value of an integer *n*. *PVSS* is infinitely differentiable and, as taking two derivatives shows, everywhere strictly concave. *PVEXACT* is infinitely differentiable and concave only within any interval which contains no integers. It is not differentiable at any integer, nor is it concave in any interval containing an integer. These departures or failures result from its left hand derivative being less than its right hand derivative on the integers, so that its graph may be described as being the graph of *PVSS*'s, but with concave arcs superimposed across the intervals between consecutive integers. As *s* increases, these arcs disappear in the limit, as illustrated in Figure 3 for $i = .04$ and $s = 40$ years.[3] In Appendix 3 we analytically compute these left- and right-hand derivatives, and relate them to the derivative of *PVSS.*

[3] The smooth function is *PVSS*. The *PVEXACT* function consists of a series of "cusps" which coincide with *PVSS* on the integers. We exaggerated the bend in the cusps in order to better illustrate the *PVEXACT* function.

Figure 3. *PVSS* and *PVEXACT* Functions with i = .04



The extension of the *PVSS* function from the integers to the real numbers follows an old tradition in mathematics: the principle of the *permanence of form*. The latter idea consists in extending the fundamental laws and operations which are applicable to positive integers to ever wider collections of numbers – here rational and all irrational numbers. Mathematicians who developed this argument include Peacock, in his *Arithmetic and Symbolic Algebra* in 1842, Hankel in his *Complexe Zahli'imysfeme* in 1867, and Cantor, extending results to the irrationals in 1871.[4]

Now, given that a function has its values prescribed on the positive integers, there are infinitely many functions which extend these values to the real numbers. Indeed, because the function $\sum_{m=1}^{m=k} b_m \sin(2\pi ms)$ for arbitrary choices of $\{b_m\}$ vanishes on integer values of s, it may be added to any extension to produce another equally valid extension. Despite the appeal of the permanence of form argument, it clearly leads to the "wrong" extension of the integral present value function since it does not coincide with the economically meaningful *PVEXACT* function.

[4] *College Algebra*, by James Harrington Boyd, Scott, Foresman and Company:1901.

As soon as we allow payments at arbitrary points in time between integers, it is natural to consider payments at all points in time between the integers, *i.e*. to consider continuous annuities. We then have

$$PVCTS(s;i) \equiv \int_{t=0}^{t=s} e^{-rt}dt = -\frac{e^{-rt}}{r}\Big|_{t=0}^{t=s} = \frac{1-e^{-rs}}{r}$$

where $e^{-r} = 1/(1+i)$ so that *r* is the continuously compounded rate of interest corresponding to the annually compounded interest rate of *i*. Multiplication of the right hand side of *PVCTS* by i/i results in $PVCTS(s;i) = \frac{i}{r}\frac{1-(1+i)^{-s}}{i} = \frac{i}{r}PVSS(s;i)$.This gives another physical interpretation of *PVSS*, in addition to that offered earlier as involving equal payments of 1 at intervals $(n+\theta)/(n+1)$ apart. Here *PVSS* is shown to correspond to payments of a continuous annuity of $(r/i) < 1$, where the latter inequality follows from:

$e^r = 1 + r + \frac{r^2}{2!} + \cdots = 1 + i$, so that $i > r$.

Of course, if the continuous annuity is 1, $i > r$ implies that *PVCTS>PVSS*.

Now, a continuous annuity speeds up the uniform payments as much as possible away from the end of the period, so we expect that, if the payment is the same, its present value will be larger. Our definition of *PVCTS* attempted to correct for this acceleration as much as possible, by taking its argument, *i*, and adjusting it downward to *r*.

Appendix 1

The difference between formula (2) and (1) is

(A1) $$(1/i)[1-(1+i)^{-n}]+\frac{\theta}{(1+i)^{n+\theta}}-(1/i)[1-(1+i)^{-(n+\theta)}] =$$

(A2) $$(1/i)[1-(1+i)^{-n}+\theta i(1+i)^{-(n+\theta)}-1+(1+i)^{-(n+\theta)}] =$$

(A3) $$(1/i)[(1+i)^{-(n+\theta)}+\theta i(1+i)^{-(n+\theta)}-(1+i)^{-n}] =$$

(A4) $$(1/i)(1+i)^{-n}[(1+i)^{-\theta}+\theta i(1+i)^{-\theta}-1] =$$

(A5) $$(1/i)(1+i)^{-n}[(1+i)^{-\theta}(1+\theta i)-1] =$$

(A6) $$(1/i)(1+i)^{-n}\{[1-\theta i+\frac{-\theta(-\theta-1)}{2}i^2+...](1+\theta i)-1\} =$$

(A7)
$$(1/i)(1+i)^{-n}\{1+\theta i-\theta i-\theta^2 i^2+\frac{\theta(\theta+1)}{2}i^2+\frac{\theta^2(\theta+1)}{2}i^3+...-1\} \approx$$

(A8) $$(1/i)(1+i)^{-n}[-\theta^2 i^2+\frac{\theta(\theta+1)}{2}i^2] =$$

(A9) $$(1/i)(1+i)^{-n}[\frac{-2\theta^2 i^2+\theta^2 i^2+\theta i^2}{2}] =$$

(A10) $$i(1+i)^{-n}[\frac{-\theta^2+\theta}{2}] =$$

(A11) $$\frac{1}{2}\left[\frac{\theta(1-\theta)}{(1+i)^n}\right]i$$ repeating text formula (5)

Steps (A2)-(A5) rearrange (A1). (A6) uses the general binomial theorem. (A7) is a rearrangement of (A6). Terms involving i^3 and higher order terms in *i* are dropped in (A8). Steps (A9)-(A11) simplify (A8). We note that (A11) approaches zero as θ approaches either zero or one (*i.e.*, the term of the annuity approaches integer values of n or $n + 1$). Also, (A11) approaches zero as n approaches infinity and as i approaches zero.

Appendix 2

This note, along with Appendix 1, has provided the general notation to explain why there is a difference in the present value depending upon whether one is using a calculator, the Excel PVSS function, or making the calculations year by year in a spreadsheet. David Jones, in his query, posed a simple data set consisting of a 4.5 year time period, a 2% interest rate, and a constant $1.00 per year payout for 4 years followed by one payment of $0.50 at year 4.5. He thereby eliminated the need to consider any growth rate in the payout. However, one could easily envision the 2% interest rate as being a Net Discount Rate (NDR) which many economists use to calculate present value and in which the growth rate and interest rate are combined so that the growth rate can be assumed to be 0%. The Table in Appendix 2 has been included so that the reader can visualize the difference as seen by Jones when he made the calculations and posed his question. These are the same variables as used in the general notation discussion in this note.

The first section of the Table contains the variables entered into a handheld calculator. Using the variables listed in the preceding paragraph, the calculator yields a present value of $4.26284. This was verified by using three calculators by three different manufacturers to insure that all are using the same formula. That formula is provided in the first section of the Table.

In the second section of the Table, the variables were entered into the PV (Present Value) function of an Excel spreadsheet (PVSS). The answer was found to be identical to that obtained using a calculator: $4.26284. The data entry sequence is different in Excel when compared to the data entry sequence in a calculator, but the underlying formulae are the same. To see the Excel formula, enter the Help section of Excel and then type PV Function in the search box.

The third and final section of the Table is a spreadsheet wherein the same variables are used to create a year by year calculation of the present value, with the value for each year calculated individually and shown in the PV column. These are then summed to arrive at a present value of $4.26510. Each PV cell in the spreadsheet contains the formula for the present value of a lump sum, and that formula is shown above the spreadsheet.

This note has established that the difference in the solutions is minor; therefore, either method could be used. As explained in the general notation discussion, this is not a constant difference and will change as the input variables change. However, we emphasize that the difference will always be very small with the year by year spreadsheet solution always being slightly higher than the calculator solution or the Excel PV (PVSS) function solution.

Handheld Calculator v. Excel PV Function (PVSS) v. Year by Year Calculation

Using Handheld Calculator

Formula =PMT*(1/i)-(1/(i*(1+i)^n))

This is the formula used in handheld calculators.

Years (n)	4.5
Interest (i)	2%
Payment/Period(PMT)	$1.00
PV	**$4.26284**

Using Excel PV Function (PVSS)

Excel PV Function =PV(D19/D23,D20*D23,D21,D22,D17)

		Col. D
Row 17	**End of Period**	0
Row 18		
Row 19	**Int**	2.0%
Row 20	**n (years)**	4.5
Row 21	**Pmt**	$1.00
Row 22	**FV**	$0.00
Row 23	**m (discounting periods/yr)**	1

PV	**$4.26284**

For explanation of formula see PV Function in Excel Help.

Using PV Formula in Year by Year Calculation

Formula for calculating the present value of a lump sum

pv = fv / (1+i)^n

This is the formula entered by user in each of the PV cells.

n	FV($)	i	PV
1	1	0.02	$0.98039
2	1	0.02	$0.96117
3	1	0.02	$0.94232
4	1	0.02	$0.92385
4.5	0.5	0.02	$0.45737

Present value = **$4.26510**

Appendix 3

This appendix quantifies and proves the inequality among the derivatives (right and left) for $PVEXACT(s;i)$ and $PVSS(s;i)$ suggested in the text, and notes a lemma connecting the continuous and discrete interest rates which ties these together. First, the Lemma. For a given discrete interest rate *i* and its continuous analogue *r*, related by $e^r = (1+i)$ we have, for $i > 0$,

$$1 - r < r/i < 1 .$$

Proof: Only the first inequality needs to be shown, since the second one has been established earlier. The first inequality is equivalent to $i < r/(1-r) = r(1+r+r^2+\cdots).$

But $e^r = 1 + r + \frac{r^2}{2!} + \cdots = 1 + i \Rightarrow\ i = r + \frac{r^2}{2!} + \cdots < r + r^2 \cdots = r/(1-r).$

We now state the result which describes the behavior of the directional derivatives at the integers in the graph.

Proposition. $PVSS'(n;i) = (r/i)(e^{-rn})$

$$PVEXACT'_+\ (n;i) = e^{-nr}$$

$$PVEXACT'_-\ (n;i) = e^{-rn}(1-r)$$

Before proving this, notice first that, from the lemma, $PVEXACT'_-\ (n;i) < PVSS'(n;i) < PVEXACT'_+\ (n;i).$ Further, all derivatives are positive, and all go to zero with increasing *n*. The proof is computation. The first derivative is easiest:

$PVSS(s;i) = (1/i)[1-(1+i)^{-s}] = (1/i)[1-e^{-rs}].$

Clearly $PVSS'(s;i) = (r/i)[e^{-rs}] = (r/i)\ 1/(1+i)^s$ everywhere, so that on the integers, for arbitrary *n*,

$PVSS'(n;i) = (r/i)[e^{-rn}] = (r/i)\ 1/(1+i)^n .$

More difficult are the right hand and left hand derivatives (note the "+"and "– " subscripts) at an integer *n*, given by definition as

$$PVEXACT'_{+}(n;i) \equiv \lim_{\theta \downarrow 0} \frac{PVEXACT(n+\theta;i) - PVEXACT(n;i)}{\theta}$$

$$PVEXACT'_{-}(n;i) \equiv \lim_{\theta \uparrow 0} \frac{PVEXACT(n+\theta;i) - PVEXACT(n;i)}{\theta}$$
$$= \lim_{\theta \downarrow 0} \frac{PVEXACT(n-\theta;i) - PVEXACT(n;i)}{-\theta}$$

Starting with the easier $PVEXACT'_{+}(n;i)$, forming the numerator in its limit involves the next two terms:

$$PVEXACT(n+\theta;i) \equiv (1/i)[1-(1+i)^{-[n+\theta]}] + \frac{(n+\theta-[n+\theta])}{(1+i)^{n+\theta}}$$
$$= (1/i)[1-(1+i)^{-n}] + \frac{\theta}{(1+i)^{n+\theta}}$$
$$PVEXACT(n;i) \equiv (1/i)[1-(1+i)^{-[n]}] + \frac{(n-[n])}{(1+i)^{n}} = (1/i)[1-(1+i)^{-n}]$$

$$PVEXACT'_{+}(n;i) \equiv \lim_{\theta \downarrow 0} \frac{1}{\theta} \frac{\theta}{(1+i)^{n+\theta}} = \frac{1}{(1+i)^{n}} = e^{-nr} > \frac{r}{i} e^{-nr}$$
$$= PVSS'(n;i)$$

Finally, the more difficult left limit:

$$PVEXACT(n-\theta;i) \equiv (1/i)[1-(1+i)^{-[n-\theta]}] + \frac{(n-\theta-[n-\theta])}{(1+i)^{n-\theta}}$$
$$= (1/i)[1-(1+i)^{-(n-1)}] + \frac{n-\theta-(n-1)}{(1+i)^{n-\theta}}$$
$$= (1/i)[1-e^{-r(n-1)}] + e^{-r(n-\theta)}(1-\theta)$$

and

$$PVEXACT(n;i) \equiv (1/i)[1-(1+i)^{-(n)}] + \frac{(n-[n])}{(1+i)^{n}} = (1/i)[1-e^{-rn}]$$

where $[n-\theta] = n-1$.

$$
\begin{aligned}
PVEXACT'_{-}(n;i) &\equiv \lim_{\theta\downarrow 0}\frac{(1/i)[1-e^{-r(n-1)}]+e^{-r(n-\theta)}(1-\theta)-(1/i)[1-e^{-rn}]}{-\theta}\\
&=\lim_{\theta\downarrow 0}\frac{(1/i)[e^{-rn}-e^{-r(n-1)}]+e^{-r(n-\theta)}(1-\theta)}{-\theta}\\
&=\lim_{\theta\downarrow 0}\left((1/i)e^{-rn}\frac{(1-e^{r})}{-\theta}+\frac{e^{-r(n-\theta)}(1-\theta)}{-\theta}\right)\\
&=\lim_{\theta\downarrow 0}\left((1/\theta)e^{-rn}\frac{(e^{r}-1)}{i}-\frac{e^{-rn}e^{r\theta}}{\theta}+e^{-rn}e^{r\theta}\right)\\
&=\lim_{\theta\downarrow 0}\left(e^{-rn}\frac{1-e^{r\theta}}{\theta}+e^{-rn}e^{r\theta}\right)\\
&=e^{-rn}\lim_{\theta\downarrow 0}\left(\frac{1-e^{r\theta}}{\theta}\right)+e^{-rn}\lim_{\theta\downarrow 0}e^{r\theta}\\
&=e^{-rn}(1-r)
\end{aligned}
$$

Of the seven equalities, the first substitutes the definition of left hand derivative and uses the fact that for $[n-\theta]=n-1$,for $\theta>0$, the second cancels $1/i$ terms, the third factors out e^{-rn}, the fourth re-groups and re-distributes, the fifth uses the previously noted continuous interest/discrete interest equality $e^{r}=1+i$ in the form $\frac{(e^{r}-1)}{i}=1$, the sixth groups two terms which individually would go to infinity, and the seventh uses the observation that $\frac{1-e^{r\theta}}{\theta}=\frac{1-(1+r\theta+(r\theta)^{2}/2!+(r\theta)^{3}/3!+\cdots)}{\theta}=-r+o(\theta)$ where $o(\theta)$ means that the terms divided by the argument θ go to 0 faster than θ (L'Hospital's Rule would work as well).