Gary R. Skoog and James E. Ciecka. 2012. Recursions in Forensic Economics. *Journal of Legal Economics* 18(2): pp. 143–160.

Recursions in Forensic Economics

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I. Introduction

This note is about recursions and two specific recursions that have become part of the forensic economics literature. We explain the concept of a recursion with clear examples and within a context most likely to be encountered by, and used by, forensic economists. The example calculations presented in this note are rich enough to illustrate essential ideas, but they do not get bogged down in unnecessary details. Three examples of recursions are proffered – the first is from mathematics, the second deals with probabilities of survival and leads to the probability mass function (pmf) for additional years of life, and the third focuses on additional years of labor market activity and ultimately the pmf for activity (Skoog and Ciecka, 2002 and 2011). This last example demonstrates the efficiency of recursions relative to other computational methods.

We begin with a definition of the term recursion.

> *A recursion is a process in which something is calculated in terms of an entity similar to itself.*

Recursive processes involve repeated evaluations of a formula or formulae typically accomplished efficiently with looping routines in computer programs. Once a recursion starts, the object of a calculation depends on previously calculated values. Performing a few hand calculations may be the best way to understand how a recursion works, but such calculations can become tedious and it may be easy to lose one's way. We have tried to minimize the likelihood of confusion arising in Example 2 and especially Example 3 by documenting each step in our examples with comments in square brackets, immediately to the right of each statement, that tell the reader the origin of the statement.

Example 1. Consider the factorial function $f(n)$ defining the product of the first n integers:

$$f(n) = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n. \tag{1}$$

The product of the first $n+1$ integers could be calculated as

$$f(n+1) = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \cdot (n+1) \tag{2}$$

by using the definition of the factorial function, but it also is possible to calculate the product of the first $n + 1$ integers with the *Factorial Recursion* defined in equation (3).

$$f(n+1) = (n+1) \cdot f(n) \tag{3}$$

The *Factorial Recursion* involves multiplying only $(n+1)$ by $f(n)$ rather than starting from the beginning and doing the entire multiplication $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \cdot (n+1)$ using equation (2). In addition to specifying a relation between two similar entities [here, $f(n)$ and $f(n + 1)$], the *Factorial Recursion* illustrates that a recursion can reduce the number of calculations or steps to reach some objective. If, for example, $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 40{,}320$ has already been calculated, it is more efficient to compute 9(40,320) using the *Factorial Recursion* defined in equation (3) than to start the calculation from the beginning and calculate $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$ with equation (2).

Example 2. Equation (3) is an example of a forward recursion because it starts with $f(2) = 2 \cdot f(1)$, moves forward to $f(3) = 3 \cdot f(2)$, to $f(4) = 4 \cdot f(3)$, and so on. However, we may know an ending value of something rather than its initial value and work backwards to earlier values. We use the following *Additional Years of Life Recursion* in (4a) and (4b) as an example of such a backward looking process.[1] This recursion's purpose is to produce pmfs for additional years of life. To this end, we use the following notation and terms.

Let BA denote the youngest exact age in a population under consideration; TA is the youngest exact age at which there are no survivors from this population. We assume that $TA = 111$, and BA can be as small as zero. Assume that transitions to death only occur at the midpoint between exact ages. Let p_x^d denote the probability that a person exact age x will die at age $x + .5$; $(1 - p_x^d)$denotes the probability of a person age x passing through the age $x + .5$ transition point and continuing to survive at age $x + 1$; y denotes years of additional life beyond age x; and $p(x, y)$ is the probability that a person exact age x survives y more years (no less and no more than y years).

The *Additional Years of Life Recursion* is defined in (4a) and (4b).

$$p(x, .5) = p_x^d \quad \text{for } x = 0, 1, \ldots, TA - 1 \tag{4a}$$

$$p(x, y) = (1 - p_x^d) p(x+1, y-1)$$
$$\text{for } x = BA, \ldots, TA - 2 \text{ and } y = 1.5, 2.5, \ldots, TA - (x + .5) \tag{4b}$$

Equation (4a) says that the probability that a person age x survives only .5 years (no less and no more) beyond age x is the probability of dying at age $x + .5$, which is p_x^d. The intuition behind recursion (4b) is as follows: $p(x + 1, y - 1)$ is the probability that a person age $x + 1$ will live $y - 1$ years (not at least $y - 1$ years but exactly $y - 1$ years) beyond age x. There would be one more year of survival (*i.e.*, y years) when measured from age x, but survival is not certain between age x and $x + 1$. Hence, we multiply the survival probability $(1 - p_x^d)$ by $p(x + 1, y - 1)$, which yields the probability of a person age x surviving y years and thereby producing recursion (4b).

To make this example more concrete, assume that $BA = 108$, and the probability of a 108-year-old person dying between ages 108 and 109 is .20; for those alive at age 109, the probability of dying between ages 109 and 110 is .60; and everyone alive at age 110 dies between ages 110 and 111. That is, $p_{108}^d = .20$, $p_{109}^d = .60$, and $p_{110}^d = 1.00$. With the midpoint transition assumption, deaths occur at ages 108.5, 109.5 and 110.5.

From (4a) and the probabilities assumed in this example, we have

$$p_{108}^d = p(108, .5) = .20 \tag{5a}$$

[The probability of a 108 year old living .5 years is .20];

$$p_{109}^d = p(109, .5) = .60 \tag{5b}$$

[The probability of a 109 year old living .5 years is .60]; and

$$p_{110}^d = p(110, .5) = 1.00 \tag{5c}$$

[The probability of a 110 year old living .5 years is 1.00].

In other words, the probability of a 108-year-old person living .5 years and thereby dying at age 108.5 with our midpoint assumption is .20; the probability of a 109 year old living .5 years and thereby dying at age 109.5 is .60; and the probability of a 110-year-old person living .5 years and then dying at age 110.5 is 1.00. So far, we do not have a pmf for years of additional life for either age 108 or 109. We merely have the first piece of the pmf for each of these ages. We do, however, have the entire pmf at age 110. It is simply $p(110, .5) = 1.00$. That is, there is a 100% probability of living .5 years at age 110. We use recursion (4b) to find the remaining piece of the pmf for years of additional life at age 109. Letting $y = 1.5$ additional years of life, we have

$$p(x, y) = (1 - p_x^d)p(x + 1, y - 1) \tag{6a}$$

[Recursion (4b)]

$$p(109, 1.5) = (1 - p_{109}^d)p(110, .5) \tag{6b}$$

[Let $y = 1.5$ and $x = 109$]

$$p(109, 1.5) = (1 - .60)(1.00) \tag{6c}$$

[Using (5b) and (5c)]

$$p(109, 1.5) = .40 \tag{6d}$$

[The probability of a 109 year old living 1.5 years is .40].

At this point, the pmf for 109-year-old people is complete since we have found $p(109, .5) = .60$ and $p(109, 1.5) = .40$ (see (5b) and (6d)), which comprise the entire pmf at age 109 and with probabilities that sum to 1.00.

We use recursion (4b) again to find the remaining pieces of the pmf for additional years of life at age 108 by letting $y = 1.5$ and then 2.5 years. These probability pieces are

$$p(x, y) = (1 - p_x^d)p(x + 1, y - 1) \tag{7a}$$

[Recursion (4b)]

$$p(108, 1.5) = (1 - p_{108}^d)p(109, .5) \tag{7b}$$

[Let $y = 1.5$ and $x = 108$]

$$p(108, 1.5) = (1 - .20)(.60) \tag{7c}$$

[Using (5a) and (5b)]

$$p(108, 1.5) = .48. \tag{7d}$$

[The probability of a 108 year old living 1.5 years is .48]; and

$$p(x, y) = (1 - p_x^d)p(x + 1, y - 1) \tag{8a}$$

[Recursion (4b)]

$$p(108, 2.5) = (1 - p_{108}^d)p(109, 1.5) \tag{8b}$$

[Let $y = 2.5$ and $x = 108$]

$$p(108, 2.5) = (1 - .20)(.40) \tag{8c}$$

[Using (5a) and (6d)]

$$p(108, 2.5) = .32 \tag{8d}$$

[The probability of a 108 year old living 2.5 years is .32];

Now, the pmf for 108-year-old people is complete. We have found $p(108, .5) = .20$, $p(108, 1.5) = .48$, and $p(108, 2.5) = .32$ (see (5a), (7d), and (8d)), which comprise the entire pmf at age 108 and with probabilities summing to 1.00.

The calculations comprising pmfs at ages 108, 109, and 110 are summarized in Table 1.

With pmfs in hand, distributional characteristics like the mean (*i.e.*, life expectancy), median, standard deviation, and skewness can easily be computed. The *Additional Years of Life Recursion* (4a) and

Table 1. Probability Mass Functions for Additional Years of Life

	Age 108	Age 109	Age 110
	$p(108, .5) = .20$	$p(109, .5) = .60$	$p(110, .5) = 1.00$
	$p(108, 1.5) = .48$	$p(109, 1.5) = .40$	
	$p(108, 2.5) = .32$		
Sum of Probabilities	1.00	1.00	1.00

4(b) could be applied to real data back as far as age zero in exactly the same manner as illustrated with the foregoing example calculations. Finally, we note that the definition of a recursion calls for something calculated in terms of an entity similar to itself as is done in recursion (4b). The left side of (4b) is $p(x, y)$, the probability of a person age x living y more years; and the right side contains $p(x+1, y-1)$, which is the probability that a one-year-older person will live one less year. These probabilities are similar objects, and recursion (4b) specifies the process that relates these probabilities. The recursion "pushes" age back one year and "adds" one more year of life with an adjustment for the probability of survival between ages x and $x+1$.

Example 3. Equations (9a)-(9d) comprise the *Additional Years of (Labor Market) Activity Recursion* or simply the *YA Recursion*.

$$p_{YA}(x, i, 0) = p_x^d + {}^i p_x^i p_{YA}(x+1, i, 0) \tag{9a}$$

$$p_{YA}(x, a, .5) = p_x^d + {}^a p_x^i p_{YA}(x+1, i, 0) \tag{9b}$$

$$p_{YA}(x, i, y) = {}^i p_x^a p_{YA}(x+1, a, y-.5) + {}^i p_x^i p_{YA}(x+1, i, y), \quad x = BA, \ldots, TA-1,\ y = 1, 2, \ldots, TA-x \tag{9c}$$

$$p_{YA}(x, a, y) = {}^a p_x^a p_{YA}(x+1, a, y-1) + {}^a p_x^i p_{YA}(x+1, i, y-.5), \quad x = BA, \ldots, TA-2,\ y = 1.5, 2.5, \ldots, TA-(x+.5) \tag{9d}$$

This recursion enables users to calculate pmfs for additional years of labor market activity; but it is more complex than the *Additional Years of Life Recursion* because two-way (*i.e.*, in and out) movements in labor market activity can occur but only one-way (*i.e.*, living to dead) movement occurs in Example 2. In the present example, $p_{YA}(x, a, y)$ denotes the probability that a person age x who is active in the labor force will accumulate y years of activity from age x until the end of life. Similarly, $p_{YA}(x, i, y)$ denotes the probability of a person age x who is inactive in the labor force will accumulate y years of activity from age x to the end of life. Assume that transitions between labor force states and to death only occur at the midpoint between exact ages. Transition probabilities between labor force states are denoted by ${}^{a}p_x^a$, ${}^{a}p_x^i$, ${}^{i}p_x^a$, ${}^{i}p_x^i$ where the left superscript is the labor force state at age x and the right superscript is the state at age $x+1$. So, for example ${}^{a}p_x^i$ means a person is active at age x and inactive at age $x+1$ with transition between states occurring at age $x+.5$. Let p_x^d denote the probability that a person exact age x will die at age $x+.5$ and assume that mortality probability does not depend on whether a person is active or inactive in the labor force; and ${}^{a}p_x^a + {}^{a}p_x^i + p_x^d = 1$ and ${}^{i}p_x^a + {}^{i}p_x^i + p_x^d = 1$. Let BA denote the youngest age in at which labor market activity can occur; TA is the youngest exact age at which there are no survivors from the population under consideration. We assume that $TA = 111$, and BA can vary depending on the educational level of the population considered.

The *YA Recursion* also includes supplementary conditions (9e)-(9i), but they are not recursive in nature. They are merely formal statements of intuitive and obvious properties and assumptions of pmfs. Condition (9e) rules out negative years of future labor market activity, and there cannot be more years of activity than there are years between age x and the age at which everyone has died. In (9f) we say that no activity or inactivity occurs at age TA since everyone has died prior to age TA. Condition, (9g) states that everyone alive at age $TA-1$ transitions to the date state. Condition (9h) expresses the assumption that mortality is independent of activity status a and i. Condition (9i) states that there is zero probability that an active person will accumulates zero years of future labor market activity. This condition holds because an active person age x cannot transition to inactivity until age $x+.5$ and thereby must be in the labor force for at least .5 years.

$$p_{YA}(x,a,y) = p_{YA}(x,i,y) = 0 \quad \text{if } y<0 \text{ or } y>TA-(x+.5). \tag{9e}$$

$$p_{YA}(TA,a,0) = p_{YA}(TA,i,0) = 1 \tag{9f}$$

$${}^{a}p_{TA-1}^{d} = {}^{i}p_{TA-1}^{d} = 1 \tag{9g}$$

$${}^{a}p_x^d = {}^{i}p_x^d = p_x^d \quad \text{for } x = BA, \ldots, TA-1 \tag{9h}$$

$$p_{YA}(x,a,0) = 0 \tag{9i}$$

The *YA Recursion* defined by (9a)-(9d) is another example of a backward recursion. The recursion works backwards in age because all labor market activity and inactivity eventually ends in death. This observation enables the recursion to start at age $TA-1$ which leads to death with certainty at age $TA - .5$. We will continue to assume that $TA = 111$ but will work backwards only to age 108 – far enough back to clearly show how the recursion works. Our numerical calculations use the following assumptions:[2]

The active-to-active transition probability between ages 108 to 109 and between ages 109 to 110 is .70, *i.e.*,

$$ {}^{a}p^{a}_{108} = {}^{a}p^{a}_{109} = .70. \tag{10a} $$

The active-to-inactive transition probability between ages 108 to 109 and between ages 109 to 110 is .20, *i.e.*,

$$ {}^{a}p^{i}_{108} = {}^{a}p^{i}_{109} = .20. \tag{10b} $$

The inactive-to-active transition probability between ages 108 to 109 and between ages 109 to 110 is .50,, *i.e.*,

$$ {}^{i}p^{a}_{108} = {}^{i}p^{a}_{109} = .50. \tag{10c} $$

The inactive-to-inactive transition probability between ages 108 to 109 and between ages 109 to 110 is .40, *i.e.*,

$$ {}^{i}p^{i}_{108} = {}^{i}p^{i}_{109} = .40. \tag{10d} $$

The probability of a transition to death between ages 108 to 109 and between ages 109 to 110 is .10 regardless of whether a person is active or inactive before dying,, *i.e.*,

$$ p^{d}_{108} = p^{d}_{109} = .10. \tag{10e} $$

Once a person reaches age 110.5, death occurs with certainty because no one lives to age 111. Therefore, the probability of death at age 110.5 is 1.00, *i.e.*, $p^{d}_{110} = 1.00$. The latter probability also implies ${}^{a}p^{a}_{110} = {}^{a}p^{i}_{110} = {}^{i}p^{a}_{110} = {}^{i}p^{i}_{110} = 0$ since the only transition at age 110 is to death and not to any labor force state.[3]

The left side of recursion (9a) is the probability that an inactive person age x will accumulate zero active time in the future. If an inactive person age x were to die at age $x + .5$, there would be no activity after age x. This event occurs with probability p^{d}_{x} which is the first term on the right side of (9a). Rather than dying at age $x + .5$, an inactive person might remain inactive to age $x + 1$ (which occurs with probability ${}^{i}p^{i}_{x}$) and accumulate no labor force time from age $x + 1$ onwards [which occurs with probability $p_{YA}(x + 1, i, 0)$]. The product of ${}^{i}p^{i}_{x}$ and $p_{YA}(x + 1, i, 0)$ is the second probability term on the right side of (9a). We can evaluate (9a) at ages 108, 109, and 110 using the assumed numerical transition probabilities in (10a)-(10e), but we work backwards from age 110 because we know how to start the recursion at age 110 as shown in (11a)-(11d) and continue it with equations

(12a)-(12d) and (13a)-(13d) for ages 109 and 108.

$$p_{YA}(x, i, 0) = p_x^d + {}^i p_x^i p_{YA}(x+1, i, 0) \quad (11a)$$

[Recursion (9a)]

$$p_{YA}(110, i, 0) = p_{110}^d + {}^i p_{110}^i p_{YA}(111, i, 0) \quad (11b)$$

[Let $x = 110$]

$$p_{YA}(110, i, 0) = 1.00 + 0(1) \quad (11c)$$

[Using (10e) and (9f)]

$$p_{YA}(110, i, 0) = 1.00 \quad (11d)$$

[The probability of an inactive 110 year old accumulating zero labor market time is 1.00.]

$$p_{YA}(x, i, 0) = p_x^d + {}^i p_x^i p_{YA}(x+1, i, 0) \quad (12a)$$

[Recursion (9a)]

$$p_{YA}(109, i, 0) = p_{109}^d + {}^i p_{109}^i p_{YA}(110, i, 0) \quad (12b)$$

[Let $x = 109$]

$$p_{YA}(109, i, 0) = .10 + .40(1.00) \quad (12c)$$

[Using (10d), (10e), and (11d)]

$$p_{YA}(109, i, 0) = .50 \quad (12d)$$

[The probability of an inactive 109 year old accumulating zero labor market time is .50.]

$$p_{YA}(x, i, 0) = p_x^d + {}^i p_x^i p_{YA}(x+1, i, 0) \quad (13a)$$

[Recursion (9a)]

$$p_{YA}(108, i, 0) = p_{108}^d + {}^i p_{108}^i p_{YA}(109, i, 0) \quad (13b)$$

[Let $x = 108$]

$$p_{YA}(108, i, 0) = .10 + .40(.50) \quad (13c)$$

[Using (10d), (10e), and (12d)]

$$p_{YA}(108, i, 0) = .30 \quad (13d)$$

[The probability of an inactive 108 year old accumulating zero labor market time is .30.]

Recursion (9b), $p_{YA}(x, a, .5) = p_x^d + {}^a p_x^i p_{YA}(x+1, i, 0)$, works the same way as (9a) but for actives. An active person at age x can accumulate only .5 years of future activity by dying between ages x and $x+1$ or turning inactive between ages x and $x+1$ and remaining inactive thereafter. These are the two probability terms on the right side of recursion (9b). We evaluate recursion (9b) below for ages 108, 109, and 110 staring with age 110 and working backwards.

$$p_{YA}(x, a, .5) = p_x^d + {}^a p_x^i p_{YA}(x+1, i, 0) \quad (14a)$$

[Recursion (9b)]

$$p_{YA}(110, a, .5) = p_{110}^d + {}^a p_{110}^i p_{YA}(111, i, 0) \quad (14b)$$

[Let $x = 110$]

$$p_{YA}(110, a, .5) = 1.00 + 0(1) \quad (14c)$$

[Using (10e) and (9f)]

$$p_{YA}(110, a, .5) = 1.00 \quad (14d)$$

[The probability of an active 110 year old accumulating .5 years of labor market time is 1.00.]

$$p_{YA}(x, a, .5) = p_x^d + {}^a p_x^i p_{YA}(x+1, i, 0) \quad (15a)$$

[Recursion (9b)]

$$p_{YA}(109, a, .5) = p_{109}^d + {}^a p_{109}^i p_{YA}(110, i, 0) \quad (15b)$$

[Let $x = 109$]

$$p_{YA}(109, a, .5) = .10 + .20(1.00) \quad (15c)$$

[Using (10b), (10e), and (11d)]

$$p_{YA}(109, a, .5) = .30 \quad (15d)$$

[The probability of an active 109 year old accumulating .5 years of labor market time is .30.]

$$p_{YA}(x, a, .5) = p_x^d + {}^a p_x^i p_{YA}(x+1, i, 0) \quad (16a)$$

[Recursion (9b)]

$$p_{YA}(108, a, .5) = p_{108}^d + {}^a p_{108}^i p_{YA}(109, i, 0) \quad (16b)$$

[Let $x = 108$]

$$p_{YA}(108, a, .5) = .10 + .20(.50) \quad (16c)$$

[Using (10b), (10e), and (12d)]

$$p_{YA}(108, a, .5) = .20 \quad (16d)$$

[The probability of an active 108 year old accumulating .5 years of labor market time is .20.]

So far, we have found the entire pmf for actives and inactives at age 110. Equation (14d) tells us that the probability is 100% that an active person age 110 will accumulate .5 years of activity, and equation (11d) says that the probability is 100% that an inactive 110-year-old person will have zero years of future activity. The probability statements $p_{YA}(110, a, .5) = 1.00$ and $p_{YA}(110, i, 0) = 1.00$ comprise the entire pmfs for actives and inactives at age 110. Equations (15d) and (16d) are the first pieces of the pmfs for ages 109 and 108 for actives accumulating only .5 years of future activity. Similarly, (12d) and (13d) give the probabilities of zero years of future activity for those initially inactive at those ages. The remaining probability mass values are defined by recursions (9c) and (9d).

The right-hand side of (9c) is the sum of two terms that contribute to the probability that a currently inactive person age x will accumulate y years of future activity: The first term ${}^{i}p_{x}^{a}p_{YA}(x+1, a, y-.5)$ is the product of two factors. $p_{YA}(x+1, a, y-.5)$ is the probability that a person active at age $x+1$ will have $y-.5$ years of future activity and, when multiplied by ${}^{i}p_{x}^{a}$, adds another half year of activity at age x. The second term ${}^{i}p_{x}^{i}p_{YA}(x+1, i, y)$ also is the product of two factors. The latter factor $p_{YA}(x+1, i, y)$ is the probability that a person inactive at age $x+1$ will have y years of future activity and, when multiplied by ${}^{i}p_{x}^{i}$, yields no additional activity at age x. The second factors in both terms aggregate sample paths resulting from remaining active for $y-.5$ years and y years from age x+1, respectively; and their multipliers ${}^{i}p_{x}^{a}$ and ${}^{i}p_{x}^{i}$ induce an additional one-half year and zero years of activity, respectively. The right-hand side of (9d) is the sum of two terms that contribute to the probability that an active person age x will accumulate y years of activity: The first term ${}^{a}p_{x}^{a}p_{YA}(x+1, a, y-1)$ is the product of two factors. $p_{YA}(x+1, a, y-1)$ is the probability that a person active at age $x+1$ will have $y-1$ years of future activity and, when multiplied by ${}^{a}p_{x}^{a}$, yields an additional one whole year of activity at age x. The second term ${}^{a}p_{x}^{i}p_{YA}(x+1, i, y-.5)$ also is the product of two factors. The latter factor$p_{YA}(x+1, a, y-.5)$ is the probability that a person inactive at age $x+1$ will have $y-.5$ years of future activity and, when multiplied by ${}^{a}p_{x}^{i}$, adds one-half year of additional activity at age x.

Recursions (9c) and (9d) are evaluated for years of activity $y = 1$ and $y = 1.5$ to get pmfs for age 109.

$$p_{YA}(x, i, y) = {}^{i}p_{x}^{a}p_{YA}(x+1, a, y-.5) + {}^{i}p_{x}^{i}p_{YA}(x+1, i, y) \tag{17a}$$

[Recursion (9c)]

$$p_{YA}(109, i, 1) = {}^{i}p_{109}^{a}p_{YA}(110, a, .5) + {}^{i}p_{109}^{i}p_{YA}(110, i, 1) \tag{17b}$$

[Let $x = 109$ and $y = 1$]

$$p_{YA}(109, i, 1) = .5(1) + .4(0) \tag{17c}$$

[Using (10c), (10d), and (9e)]

$$p_{YA}(109, i, 1) = .50 \quad (17d)$$

[The probability of an inactive 109 year old accumulating 1 year of labor market time is .50.]

$$p_{YA}(x, a, y) = {}^a p_x^a p_{YA}(x+1, a, y-1) + {}^a p_x^i p_{YA}(x+1, i, y-.5) \quad (18a)$$

[Recursion (9d)]

$$p_{YA}(109, a, 1.5) = {}^a p_{109}^a p_{YA}(110, a, .5) + {}^a p_{109}^i p_{YA}(110, i, 1) \quad (18b)$$

[(Let $x = 109$ and $y = 1.5$]

$$p_{YA}(109, a, 1.5) = .70(1) + .20(0) \quad (18c)$$

[Using (10a), (10b), and (9e)]

$$p_{YA}(109, a, 1.5) = .70 \quad (18d)$$

[The probability of an active 109 year old accumulating 1.5 years of labor market time is .70.]

The pmfs for actives and inactives at age 109 are now complete. For actives we have $p_{YA}(109, a, .5) = .30$ and $p_{YA}(109, a, 1.5) = .70$ from (15d) and (18d), respectively. The pmf for 109-year-old inactives consists of $p_{YA}(109, i, 0) = .50$ and $p_{YA}(109, i, 1) = .50$ from (12d) and (17d), respectively.

Recursions (9c) and (9d) are evaluated for years of activity $y = 1$, $y = 1.5$, $y = 2$, and $y = 2.5$ to get pmfs for age 108.

$$p_{YA}(x, i, y) = {}^i p_x^a p_{YA}(x+1, a, y-.5) + {}^i p_x^i p_{YA}(x+1, i, y) \quad (19a)$$

[Recursion (9c)]

$$p_{YA}(108, i, 1) = {}^i p_{108}^a p_{YA}(109, a, .5) + {}^i p_{108}^i p_{YA}(109, i, 1) \quad (19b)$$

[Let $x = 108$ and $y = 1$]

$$p_{YA}(108, i, 1) = .50(.30) + .40(.50) \quad (19c)$$

[Using (10c), (10d), (15d), and (17d)]

$$p_{YA}(108, i, 1) = .35 \quad (19d)$$

[The probability of an inactive 108 year old accumulating 1 year of labor market time is .35.]

$$p_{YA}(x, a, y) = {}^a p_x^a p_{YA}(x+1, a, y-1) + {}^a p_x^i p_{YA}(x+1, i, y-.5) \quad (20a)$$

[Recursion (9d)]

$$p_{YA}(108, a, 1.5) = {}^a p_{108}^a p_{YA}(109, a, .5) + {}^a p_{108}^i p_{YA}(109, i, 1) \quad (20b)$$

[Let $x = 108$ and $y = 1.5$]

$$p_{YA}(108, a, 1.5) = .70(.30) + .20(.50) \tag{20c}$$
[Using (10a), (10b), (15d), and (17d)]

$$p_{YA}(108, a, 1.5) = .31 \tag{20d}$$
[The probability of an active 108 year old accumulating 1.5 years of labor market time is .31.]

$$p_{YA}(x, i, y) = {}^{i}p_{x}^{a} p_{YA}(x + 1, a, y - .5) + {}^{i}p_{x}^{i} p_{YA}(x + 1, i, y) \tag{21a}$$
[Recursion (9c)]

$$p_{YA}(108, i, 2) = {}^{i}p_{108}^{a} p_{YA}(109, a, 1.5) + {}^{i}p_{108}^{i} p_{YA}(109, i, 2) \tag{21b}$$
[Let $x = 108$ and $y = 2$]

$$p_{YA}(108, i, 2) = .50(.70) + .40(0) \tag{21c}$$
[Using (10c), (10d), (18d), and (9e)]

$$p_{YA}(108, i, 2) = .35 \tag{21d}$$
[The probability of an inactive 108 year old accumulating 2 years of labor market time is .35.]

$$p_{YA}(x, a, y) = {}^{a}p_{x}^{a} p_{YA}(x + 1, a, y - 1) + {}^{a}p_{x}^{i} p_{YA}(x + 1, i, y - .5) \tag{22a}$$
[Recursion (9d)]

$$p_{YA}(108, a, 2.5) = {}^{a}p_{108}^{a} p_{YA}(109, a, 1.5) + {}^{a}p_{108}^{i} p_{YA}(109, i, 2) \tag{22b}$$
[Let $x = 108$ and $y = 2.5$]

$$p_{YA}(108, a, 2.5) = .70(.70) + .20(0) \tag{22c}$$
[Using (10a), (10b), (18d), (9e)]

$$p_{YA}(108, a, 2.5) = .49 \tag{22d}$$
[The probability of an active 108 year old accumulating 2.5 years of labor market time is .49.]

The pmf for actives at age 108 consists of $p_{YA}(108, a, .5) = .20$, $p_{YA}(108, a, 1.5) = .31$, $p_{YA}(108, a, 2.5) = .49$ from (16d), (20d), and (22d). The pmf for inactives at age 108 is $p_{YA}(108, i, 0) = .30$, $p_{YA}(108, i, 1) = .35$, and $p_{YA}(108, i, 2) = .35$ from (13d), (19d), and (21d).

We summarize results for actives and inactive below in Table 2.

Recursions based on actual transition probabilities work in exactly the same manner as described in this example but go back to beginning age as young as $BA = 16$ rather than 108 used in this example. Skoog and Ciecka (2001) and Skoog, Ciecka, and Krueger (2011) calculate distributional characteristics like the mean (*i.e.*,

Table 2. Probability Mass Functions for Additional Years of of Labor Force Activity for Actives and Inactives

	Age 108	Age 109	Age 110
Actives			
	$p(108, a, .5) = .20$	$p(109, a, .5) = .30$	$p(110, a, .5) = 1.00$
	$p(108, a, 1.5) = .31$	$p(109, a, 1.5) = .70$	
	$p(108, a, 2.5) = .49$		
Sum of Probabilities	1.00	1.00	1.00
Inactives			
	$p(108, i, 0) = .30$	$p(109, i, 0) = .50$	$p(110, i, 0) = 1.00$
	$p(108, i, 1) = .35$	$p(109, i, 1) = .50$	
	$p(108, i, 2) = .35$		
Sum of Probabilities	1.00	1.00	1.00

worklife expectancy), median, standard deviation, skewness, kurtosis, and percentile points from pmfs based on recursions (9a)-(9d).

Finally, we note that recursions (9a)-(9d) are more complex than the recursions in Example 1 or Example 2. The right sides of each of the recursions (9a)-(9d) contain transition probabilities at age x which describe movement to age $x+1$. In addition, all of the probabilities for years of activity on the right sides of (9a)-(9d) are indexed to age $x+1$.

II. Computational Efficiency of Recursions

We refer to each evaluation of a recursion equation as one replication of the recursion. Each replication consists of one multiplication and an addition in the case of recursions (9a) and (9b) and two multiplications and an addition for recursions (9c) and (9d). In general, we show in the Appendix that pmfs are computed with

$$2(TA - BA) + (TA - 1 - BA)(TA - BA) \tag{23}$$

replications of recursions (9a)-(9d). In Example 3, $TA = 111$ and $BA = 108$, implying

$$\begin{aligned} &2(TA - BA) + (TA - 1 - BA)(TA - BA) \\ &\quad = 2(111 - 108) + (111 - 1 - 108)(111 - 108) \\ &\quad = 2(3) + (2)(3) = 12 \end{aligned}$$

replications consisting of the calculations in equations numbered (11) through (22).[4]

Skoog and Ciecka (2001) and Skoog, Ciecka, and Krueger (2011) assume $TA = 111$ and $BA = 16$ as the youngest beginning age for labor force activity. Evaluating (23) with these values, produces

$$\begin{aligned} &2(TA - BA) + (TA - 1 - BA)(TA - BA) \\ &\quad = 2(111 - 16) + (111 - 1 - 16)(111 - 16) = 9,120 \end{aligned} \tag{24}$$

total replications to find all the pmfs for all ages between $BA = 16$ and $TA - 1 = 110$. These calculations are done practically instantaneously on a very basic laptop computer. In the Appendix, we also show that tracing labor force activity/inactivity/death paths results in

$$2(2^{TA-BA} - 1) = 2(2^{111-16} - 1) \doteq 7.92 \times 10^{28} \tag{25}$$

paths, which are impossible to trace in any practical sense and which are a factor of 8.68×10^{24} larger than the 9,120 replications required using recursions (9a)-(9d).[5] The remarkable power of the recursions (9a)-(9d) becomes evident when we compare 9,120 simple calculations with the alternative of tracing 7.92×10^{28} labor force paths. In addition to the mind numbing number of paths to search, there are three other reasons complicating tracing paths: (1) Paths can be very long and involve as many as 95 multiplications which would be burdensome and lead to inaccuracies. (2) Even if every path were evaluated, probabilities associated with like years of activity would have to be aggregated to get pmfs. (3) The whole exercise would have to be repeated for age 17, then repeated for age 18, and so on; or, alternatively, a computer would have to store an incredibly large number of calculations.

References

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Endnotes

1. There is no unambiguous and intuitive adjectival modifier to describe a recursion. In the Example 2 recursion, a probability associated with a person's current age depends on a probability indexed to a future age and, in that sense, it is also intuitive to call it a forward recursion. In calling this a backward recursion, we have emphasized the idea that the recursion begins at an age immediately prior to the age at which everyone has died; and then the recursion works backwards to successively earlier ages.

2. For simplicity this note assumes that transitions from active to active occur with the same probability between ages 108 to 109 and between ages 109 to 110. This note makes similar constancy assumptions for active to inactive, inactive to active, and inactive to inactive transitions between ages 108 to 109 and 109 to 110. Of course, transition probabilities need not be constant from age to age.

3. The reader should keep in mind that our assumed transition probabilities into activity are unrealistically high for the age range 108–111, but we make these numerical assumptions in order to avoid working with very small probabilities

4. Each of the equations numbered (11)-(22) has four lines labeled (a) through (d). However, these lines are for expository purposes. Each set of four equations referenced (a) through (d) consists of one replication.

5. The world's fastest computes do 20,000 trillion calculations per second (standard.net/topics/business/2011/03/07). At that speed, it would take $7.92 \times 10^{28} / (20{,}000 \times 10^{12} \times 60 \times 60 \times 24 \times 365 \doteq 125{,}000$ years to search all labor force paths.

Appendix

Only inactives can accumulate zero years of future activity as shown in recursion (9a). There are

$$TA - BA \tag{A1}$$

replications of this recursion – one replication for each age $TA - 1$, $TA - 2, \ldots, BA$. The probability of .5 years of activity for initial actives is determined by recursion (9b) which also is replicated $TA - BA$ times, once for each age $TA - 1$, $TA - 2, \ldots, BA$. Thus, the combined number of replications of recursions (9a) and (9b) is

$$2(TA - BA) \tag{A2}$$

In regard to recursions (9c) and (9d), first consider age $TA - 1$. Possible values for years of activity (YA) are 0, .5 [1 year is not possible because everyone is dead at age $(TA - 1) + 1 = TA$]. Recursion (9a) accounts for 0 years and recursion (9b) for .5 years, leaving *nothing* to be calculated with (9c) and (9d). At age $TA - 2$, YA can take on four possible values 0, .5, 1, 1.5. Recursions (9a) and (9b) account for 0 and .5 years, respectively, leaving only *two* YA values (1, 1.5) to calculate with recursions (9c) and (9d). At age $TA - 3$, YA can take on six possible values 0, .5, 1, 1.5, 2, 2.5. Recursions (9a) and (9b) account for 0 and .5 years, respectively, leaving *four* YA values (1, 1.5, 2, 2.5) to calculate with recursions (9c) and (9d). At age $TA - 4$, YA can only take on eight possible values 0, .5, 1, 1.5, 2, 2.5, 3, 3.5. Recursions (9a) and (9b) account for 0 and .5 years, respectively, leaving *six* YA values (1, 1.5, 2, 2.5, 3, 3.5) to calculate with recursions (9c) and (9d). By continuing in this manner back to age BA, recursions (9c) and (9d) account for the $2(TA - 1 - BA)$ YA values (1, 1.5, 2, 2.5, …, $TA - BA - .5$). The sum of all replications of recursions (9c) and (9d) is the sum of the even integers from zero to $2(TA - 1 - BA)$ as shown in (A3).

$$\begin{aligned} S &= 0 + 2 + 4 + 6 + \cdots + 2(TA - 1 - BA) \\ &= (TA - 1 - BA)(TA - BA). \end{aligned} \tag{A3}$$

Combining (A2) and (A3), there are

$$2(TA - BA) + (TA - 1 - BA)(TA - BA) \tag{A4}$$

total replications. Evaluating (A4) with $TA = 111$ and $BA = 16$, gives

$$\begin{aligned} &2(TA - BA) + (TA - 1 - BA)(TA - BA) \\ &\quad = 2(111 - 16) + (111 - 1 - 16)(111 - 16) = 9,120 \end{aligned} \tag{A5}$$

total replications to find all the pmfs for all ages between BA and $TA -$

1 with each replication consisting of only a couple of multiplications and addition of two numbers.

In principle, labor force paths could be traced to find pmfs (Skoog and Ciecka, 2011). Consider a person who is active at age $TA - 1$ for which there is only one labor force path because the only transition is to the death state. This path is $a \rightarrow d$; there is only $2^{TA-(TA-1)} - 1 = 2^1 - 1 = 1$ such path. Possible paths for a person active at age $TA - 2$ are $a \rightarrow a \rightarrow d$, $a \rightarrow d$, and $a \rightarrow i \rightarrow d$; a total of $2^{TA-(TA-2)} - 1 = 2^2 - 1 = 3$ paths. Possible paths for a person active at age $TA - 3$ are $a \rightarrow a \rightarrow a \rightarrow d$, $a \rightarrow a \rightarrow i \rightarrow d$, $a \rightarrow i \rightarrow a \rightarrow d$, $a \rightarrow i \rightarrow i \rightarrow d$, $a \rightarrow a \rightarrow d$, $a \rightarrow i \rightarrow d$, and $a \rightarrow d$; a total of $2^{TA-(TA-3)} - 1 = 2^3 - 1 = 7$ paths. Continuing with the same reasoning, the total of all possible paths for actives at age BA is

$$2^{TA-BA} - 1. \tag{A6}$$

There are an equal number of paths for those who are initially inactive. Therefore, the combined number of paths for initial actives and inactives is

$$2(2^{TA-BA} - 1). \tag{A7}$$

With $TA = 111$ and $BA = 16$ as the youngest beginning age for labor market activity, (A7) evaluates as

$$2(2^{TA-BA} - 1) = 2(2^{111-16} - 1) \doteq 7.92 \times 10^{28}. \qquad \text{(A7)}$$